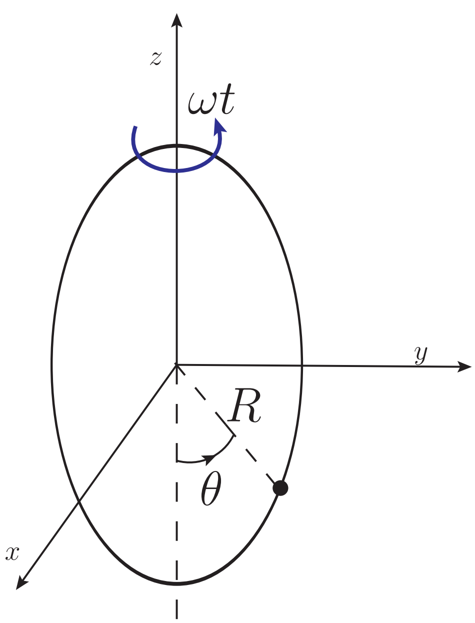
PHY 5246: Theoretical Dynamics, Fall 2015

Assignment # 2, Solutions

1 Graded Problems

Problem 1

(1.a)



The coordinates can be written using spherical coordinates as:

$$\begin{aligned} x &= R\sin\theta\cos(\omega t)\ , \\ y &= R\sin\theta\sin(\omega t)\ , \\ z &= -R\cos\theta\ , \end{aligned} \tag{1}$$

where we have used the constraints: $r = R$ and $\phi = \omega t$ (i.e. $\dot\phi = \omega$), and reduced the number of generalized coordinates to just one (θ) and its velocity ($\dot\theta$). The problem is therefore equivalent to a one-dimensional problem. Notice that the constraints are time dependent, which hint to the fact that the mechanical energy of the system may not be conserved. The kinetic and potential energy of the system are

$$\begin{aligned} T &= \frac{1}{2}m(R^2\dot\theta^2 + R^2\sin^2\theta\omega^2)\ , \\ V &= -mgR\cos\theta\ , \end{aligned} \tag{2}$$

and the Lagrangian of the system is

$$L = T - V = \frac{1}{2}m(R^2\dot\theta^2 + R^2\sin^2\theta\omega^2) + mgR\cos\theta\ . \tag{3}$$

The Euler-Lagrange equation of motion is

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} - \frac{\partial L}{\partial\theta} = 0 \longrightarrow \ddot\theta = \omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta\ . \tag{4}$$

Using the equation of motion one can indeed verify that dE/dt is not zero, as expected.

(1.b)

The equilibrium points are defined as the points where the bead, if placed there (i.e. with zero velocity), does not move. They are given therefore by the condition

$$\ddot\theta = 0 \quad\longrightarrow\quad \sin\theta\left(\omega^2\cos\theta - \frac{g}{R}\right) = 0\ , \tag{5}$$

and are

$$\theta_0 = 0, \pi \ , \ \text{and} \ \ \theta_0 = \arccos\left(\frac{g}{R\omega^2}\right) \ \ \text{if} \ \ \frac{g}{R\omega^2} < 1 \ . \tag{6}$$

Let us notice that we could get to the same result by interpreting the r.h.s of Eq. (4) as a force, $F(\theta)$ and defining an *effective potential*, $V_{\rm eff}(\theta)$ such that

$$\begin{aligned} F(\theta) &= -\frac{\partial V_{\rm eff}(\theta)}{\partial \theta} \\ V_{\rm eff}(\theta) &= -\frac{1}{2} m R^2 \sin^2\theta \omega^2 - mgR\cos\theta \ . \end{aligned}$$

Then, the equilibrium condition is equivalent to the condition that gives the extrema of $V_{\rm eff}(\theta)$, i.e.

$$\ddot{\theta} = 0 \ \ \longrightarrow \ \ F(\theta) = 0 \ \ \longrightarrow \ \ \frac{\partial V_{\rm eff}(\theta)}{\partial \theta} = 0 \ . \tag{7}$$

The nature of the equilibrium points (stable vs unstable) can then be determined by looking at the second derivative of $V_{\rm eff}(\theta)$,

$$\frac{\partial^2 V_{\rm eff}(\theta)}{\partial \theta^2} = -mR^2\omega^2\cos^2\theta + mR^2\omega^2\sin^2\theta + mgR\cos\theta \ , \tag{8}$$

at each equilibrium point. We find that $\theta_0 = \pi$ is always an unstable equilibrium point, since:

$$\left.\frac{\partial^2 V_{\rm eff}(\theta)}{\partial \theta^2}\right|_{\theta_0 = \pi} = -mR^2\omega^2 - mgR < 0 \ , \tag{9}$$

while the nature of $\theta_0 = 0$ and $\theta_0 = \arccos\left(\frac{g}{R\omega^2}\right)$ depends on the ratio $g/(R\omega^2)$. Indeed,

$$\left.\frac{\partial^2 V_{\rm eff}(\theta)}{\partial \theta^2}\right|_{\theta_0 = 0} = mR^2\omega^2\left(\frac{g}{R\omega^2} - 1\right) \tag{10}$$

and $\theta_0 = 0$ is a stable equilibrium point if $g/(R\omega^2) > 1$ while it is unstable if $g/(R\omega^2) < 1$. Viceversa,

$$\left.\frac{\partial^2 V_{\rm eff}(\theta)}{\partial \theta^2}\right|_{\theta_0 = \arccos\left(\frac{g}{R\omega^2}\right)} = mR^2\omega^2\left[1 - \left(\frac{g}{R\omega^2}\right)^2\right] \ , \tag{11}$$

and $\theta_0 = \arccos(g/(R\omega^2))$ is a stable equilibrium point if $g/(R\omega^2) < 1$, while it is unstable if $g/(R\omega^2) > 1$. The shape of $V_{\rm eff}(\theta)$ therefore changes in going from $g/(R\omega^2) > 1$ to $g/(R\omega^2) < 1$. In the first case $V_{\rm eff}(\theta)$ has a minimum at $\theta_0 = 0$ and no other extrema except a maximum at $\theta_0 = \pi$, while in the second case, $\theta_0 = 0$ becomes a maximum (as well as $\theta_0 = \pi$) and a new minimum develops at $\theta_0 = \arccos(g/(R\omega^2))$. Of course the potential is symmetric with respect to θ, so an analogous discussion holds for θ between π and 2π. We can actually say that for $g/(R\omega^2) < 1$ there are two stable equilibrium points, one to the right and one to the left of the $\theta = 0$ position, both at an angle $\theta_0 = \arccos(g/(R\omega^2))$ from the vertical.

(1.c)

To find the frequency of small oscillations about the stable equilibrium positions (θ_0) we consider a small displacement from θ_0 (let's call it δ), i.e. we write

$$\theta = \theta_0 + \delta \ , \tag{12}$$

and plug it into the equation of motion. We then expand the equation of motion in δ and its derivatives, keeping only up to linear terms, i.e. we write

$$\begin{aligned}
\sin(\theta_0+\delta) &= \sin\theta_0\cos\delta+\cos\theta_0\sin\delta=\sin\theta_0(1+O(\delta^2))+\cos\theta_0(\delta+O(\delta^3)\ , \\
\cos(\theta_0+\delta) &= \cos\theta_0\cos\delta-\sin\theta_0\sin\delta=\cos\theta_0(1+O(\delta^2))-\sin\theta_0(\delta+O(\delta^3)\ ,
\end{aligned} \tag{13}$$

and we get the equation of motion for small oscillations about θ_0 in the form

$$\ddot{\delta}=\left(\omega^2\sin\theta_0\cos\theta_0-\frac{g}{R}\sin\theta_0\right)+\delta\left[\omega^2\left(\cos^2\theta_0-\sin^2\theta_0\right)-\frac{g}{R}\cos\theta_0\right]\ . \tag{14}$$

The terms in the first parenthesis cancel because θ_0 is a solution of $\ddot{\theta}=0$ (or equivalently a minimum of V_{eff} and therefore satisfies the extreme condition in Eq. (7)), and the equation for δ reduces to

$$\ddot{\delta}+\Omega^2\delta=0\ , \tag{15}$$

which is the equation of a one-dimensional harmonic oscillator with frequency

$$\Omega^2=\frac{g}{R}\cos\theta_0-\omega^2\left(\cos^2\theta_0-\sin^2\theta_0\right)\ . \tag{16}$$

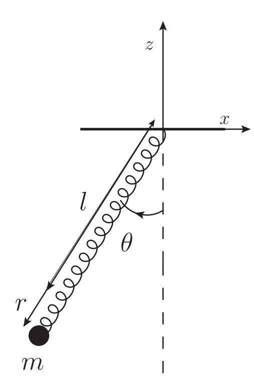
We then have two cases:

- $g/(R\omega^2)>1$, the stable equilibrium point is $\theta_0=0$ and the frequency of small oscillation about θ_0 is
$$\Omega=\left(\frac{g}{R}-\omega^2\right)^{1/2}\ , \tag{17}$$
- $g/(R\omega^2)<1$, the stable equilibrium point is $\theta_0=\arccos(g/(R\omega^2))$ and the frequency of small oscillation about θ_0 is
$$\Omega=\omega\sin\theta_0=\omega\left(1-\frac{g^2}{R^2\omega^4}\right)^{1/2}\ . \tag{18}$$

At the critical value of $\omega^2=g/R$, the system is transitioning between the two cases, and the critical values all occur at $\theta=0$.

Problem 2

The system has two degrees of freedom and given the symmetry of the problem, it is natural to choose polar coordinates, r and θ. Indeed, we will denote by r the elongation of the spring from its unscratched length. They are the generalized coordinates of the problem, in terms of which we can write the kinetic and potential energy of m as



$$\begin{aligned}
T &= \frac{1}{2}m(\dot{r}^2+(l+r)^2\dot{\theta}^2)\ , \\
V &= -mg(l+r)\cos\theta+\frac{1}{2}kr^2\ ,
\end{aligned} \tag{19}$$

and the Lagrangian of the system as

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + (l+r)^2\dot{\theta}^2) + mg(l+r)\cos\theta - \frac{1}{2}kr^2 \ . \qquad (20)$$

The equation of motion for r is then

$$m(\ddot{r} - (r+l)\dot{\theta}^2) = mg\cos\theta - kr \ , \qquad (21)$$

while the equation of motion for θ is

$$m[(l+r)\ddot{\theta} + 2\dot{r}\dot{\theta}] = -mg\sin\theta \ . \qquad (22)$$

Notice that Eq. (21) and (22) are of the form $ma_r = F_r$ and $ma_\theta = F_\theta$, where (a_r, F_r) and (a_θ, F_θ) are the components of the acceleration and of the total applied force in the radial and tangent direction respectively.

Problem 3 (1.14 of Goldstein's book)

Notice that the problem is in 3d. So, imagine the center of mass of the system to move in a circle in the (x, y) plane, while the rod+masses can also move in z.

The kinetic energy (K.E.) of the system is the K.E. of the CM (with mass $2m$) plus the K.E. of the two masses rotating with respect to the CM.

$$\begin{aligned}
T &= T_{CM} + T_{1+2 \ wrt \ CM} \\
T_{CM} &= \frac{1}{2}(2m)a^2\dot{\alpha}^2 \\
T_{1+2 \ wrt \ CM} &= 2\frac{1}{2}m\left(\frac{l}{2}\right)^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2).
\end{aligned}$$

Here the coordinates of 1 and 2 with respect to the CM are

$$x_1 = \frac{l}{2}\cos\phi\sin\theta \qquad x_2 = -\frac{l}{2}\cos\phi\sin\theta \qquad (23)$$

$$y_1 = \frac{l}{2}\sin\phi\sin\theta \qquad y_2 = -\frac{l}{2}\sin\phi\sin\theta \qquad (24)$$

$$z_1 = \frac{l}{2}\cos\theta \qquad z_2 = -\frac{l}{2}\cos\theta \qquad (25)$$

Problem 4 (1.21 of Goldstein's book)

Here we use generalized coordinates $\{r, \theta\}$ and set the length of the string to be fixed, l.

$$\begin{aligned}
T &= \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}m_2\dot{r}^2 \\
V &= -m_2g(l-r) \\
L &= T - V = \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2}m_1r^2\dot{\theta}^2 + m_2g(l+r).
\end{aligned}$$

The equation of motion for r from the Euler-Lagrange equations is

$$(m_1+m_2)\ddot{r} - m_1r\dot{\theta}^2 - m_2g = 0.$$

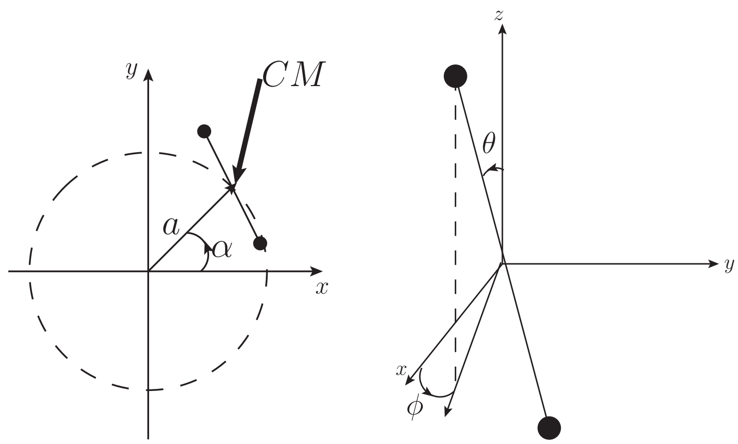
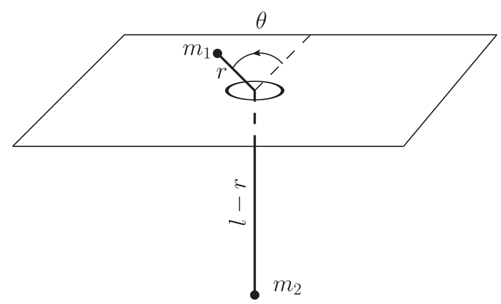


Figure 1: Figures for Problem 3



For the θ equation we note that since the Lagrangian is not a function of θ we get

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0.$$

This implies that the angular momentum of m_1 about the origin is conserved,

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m_1 r^2 \dot{\theta} = \text{constant} \Rightarrow \dot{\theta} = \frac{p_\theta}{m_1 r^2}.$$

Thus we can rewrite the r equation of motion as

$$(m_1 + m_2)\ddot{r} - \frac{p_\theta^2}{m_1 r^3} + m_2 g = 0,$$

which is now just a 1d problem.

2 Non-graded Problems

Problem 5 (1.15 of Goldstein's book)

$$U(\mathbf{r}, \mathbf{v}) = V(r) + \boldsymbol{\sigma} \cdot \mathbf{L}$$

(a)

$$
\begin{aligned}
U(\mathbf{r},\mathbf{v}) &= V(r)+\boldsymbol{\sigma}\cdot(\mathbf{r}\times m\mathbf{v}) \\
&= V(r)+m\left[\sigma_x(yv_z-zv_y)+\sigma_y(zv_x-xv_z)+\sigma_z(xv_y-yv_x)\right] \\
F_x &= -\frac{\partial V}{\partial x}+\frac{d}{dt}\frac{\partial V}{\partial \dot{x}} \\
&= -\frac{\partial V}{\partial r}\frac{x}{r}-m(-\sigma_y v_z+\sigma_z v_y)+m(\sigma_y v_z-\sigma_z v_y) \\
&= -\frac{\partial V}{\partial r}\frac{x}{r}+2m(\sigma_y v_z-\sigma_z v_y) \\
F_y &= -\frac{\partial V}{\partial r}\frac{y}{r}+2m(\sigma_z v_x-\sigma_x v_z) \\
F_z &= -\frac{\partial V}{\partial r}\frac{z}{r}+2m(\sigma_x v_y-\sigma_y v_z) \\
\mathbf{F} &= -\frac{\partial V}{\partial r}\hat{\mathbf{r}}+2m(\boldsymbol{\sigma}\times\mathbf{v})
\end{aligned}
$$

Now putting these equations into spherical coordinates and choosing $\boldsymbol{\sigma}=\sigma\hat{\mathbf{z}}$, we can plug $U(\mathbf{r},\mathbf{v})$ into the general Lagrange equations

$$
Q_i=-\frac{dU}{dx^i}+\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}^i}\right) \quad .
$$

We find the following:

$$
\begin{aligned}
U &= V(r)+m\sigma r^2\sin^2\theta\dot{\phi} \\
Q_r &= -\frac{dV}{dr}-2m\sigma r\sin^2\theta\dot{\phi} \\
Q_\theta &= -2m\sigma r^2\sin\theta\cos\theta\dot{\phi} \\
Q_\phi &= 2m\sigma r^2\sin\theta\cos\theta\dot{\theta}+2m\sigma r\dot{r}\sin^2\theta.
\end{aligned}
$$

(b)

Putting σ in the $\hat{\mathbf{z}}$ direction it is easy (but rather lengthy) to show the result

$$
Q_j=\sum_i \mathbf{F}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_j} \quad .
$$

(c)

$$
T=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2),
$$

where the equations of motion are

$$
\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}-\frac{\partial T}{\partial q_j}=Q_j.
$$

Problem 6 (1.18 of Goldstein's book)

$$L = \frac{m}{2}(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2) - \frac{k}{2}(ax^2 + 2bxy + cy^2).$$

Equations of motion:

$$\begin{aligned} m(a\ddot{x} + b\ddot{y}) + k(ax + by) &= 0 \\ m(c\ddot{y} + b\ddot{x}) + k(cy + bx) &= 0 \end{aligned}$$

- *Case 1*, $a = c = 0$.

  $$\begin{aligned} \ddot{y} + \omega^2 y &= 0 \\ \ddot{x} + \omega^2 x &= 0 \end{aligned}$$

  where $\omega = \sqrt{k/m}$. So we have two decoupled 2d harmonic oscillators with the same frequency.

- *Case 2*, $b = 0,\ c = -a$.

  $$\begin{aligned} \ddot{x} + \omega^2 x &= 0 \\ \ddot{y} + \omega^2 y &= 0. \end{aligned}$$

  and the result is the same as in the first case.

If we make a change of variables

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{array}{rccc} u = & ax & + & by \\ v = & bx & + & cy \end{array}$$

the system decouples at the level of the Lagrangian. The $b^2 - ac \neq 0$ condition is the condition for the transformation matrix to not be singular.

Problem 7 (1.22 of Goldstein's book)

$$\begin{cases} x_1 = l_1 \sin\theta_1 \\ y_1 = -l_1 \cos\theta_1 \end{cases} \qquad \begin{cases} \dot{x}_1 = l_1\dot{\theta}_1 \cos\theta_1 \\ \dot{y}_1 = l_1\dot{\theta}_1 \sin\theta_1 \end{cases}$$

$$\begin{cases} x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 \\ y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2 \end{cases} \qquad \begin{cases} \dot{x}_2 = l_1\dot{\theta}_1 \cos\theta_1 + l_2\dot{\theta}_2 \cos\theta_2 \\ \dot{y}_2 = l_1\dot{\theta}_1 \sin\theta_1 + l_2\dot{\theta}_2 \sin\theta_2 \end{cases}$$

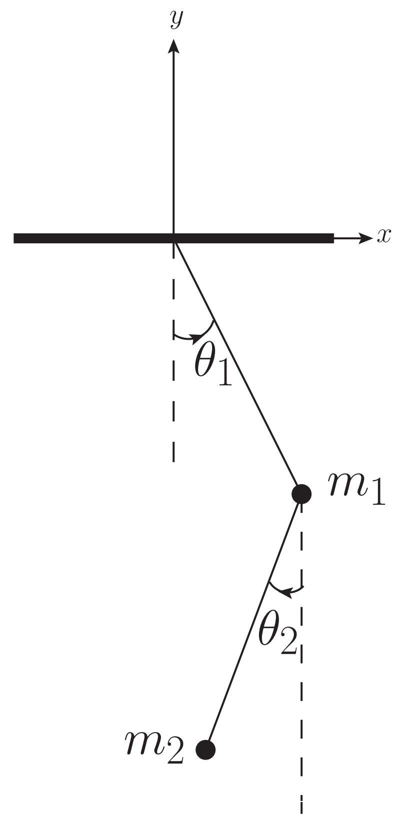
The Lagrangian is

$$\begin{aligned} L &= T_1 + T_2 - V_1 - V_2 = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) - V_1 - V_2 \\ &= \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)] + m_1 g l_1 \cos\theta_1 + m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2). \end{aligned}$$

The equations of motion (via Euler-Lagrange) are

$$(m_1 + m_2)l_1\ddot{\theta}_1 = -(m_1 + m_2)g\sin\theta_1 - m_2 l_2\cos(\theta_1 - \theta_2)\ddot{\theta}_2 - \dot{\theta}_2^2 m_2 l_2 \sin(\theta_1 - \theta_2).$$

$$m_2 l_2\ddot{\theta}_2 = -m_2 g\sin\theta_2 - m_2 l_1\ddot{\theta}_1\cos(\theta_1 - \theta_2) + \dot{\theta}_1^2 m_2 l_1\sin(\theta_1 - \theta_2).$$



y
x
θ_1
m_1
θ_2
m_2