PHY 5246: Theoretical Dynamics, Fall 2015

Assignment # 3, Solutions

1 Graded Problems

Problem 1

(1.a)

We use cylindrical coordinates and notice that $z = r\cot\alpha$. We use $\{r, \theta\}$ as generalized coordinates and write the kinetic energy of the bead as

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) = \frac{1}{2}m\left[\frac{1}{\sin^2\alpha}\dot{r}^2 + r^2\dot{\theta}^2\right] \ , \tag{1}$$

and its potential energy as

$$V = mgz = mgr\cot\alpha \ , \tag{2}$$

where we have assumed $V = 0$ at $z = 0$.

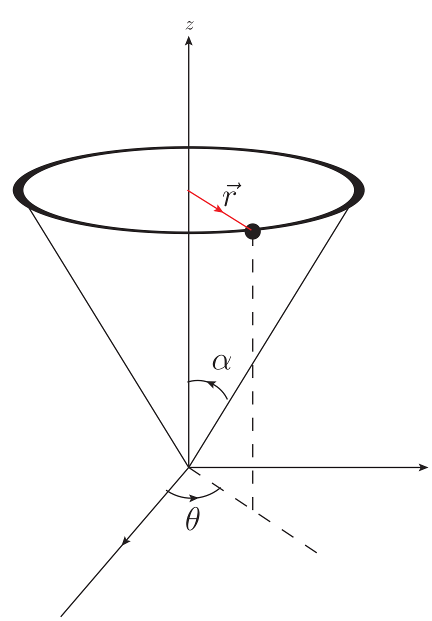
The Euler-Lagrange equation of motion for the r coordinate is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0 \quad \longrightarrow \quad \ddot{r} - r\sin^2\alpha\dot{\theta}^2 + g\sin\alpha\cos\alpha = 0 \ , \tag{3}$$

while the Euler-Lagrange equation of motion for θ is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \quad \longrightarrow \quad \frac{d}{dt}(mr^2\dot{\theta}) = 0 \quad \longrightarrow \quad l_\theta = mr^2\dot{\theta} = \text{constant} \ , \tag{4}$$

which expresses the conservation of angular momentum about the z axis.



(1.b)

If the bead is in equilibrium at $r = r_0$ then

$$\dot{\theta}|_{r=r_0} = \omega = \frac{l_\theta}{mr_0^2} \ , \tag{5}$$

since l_θ is constant (see Eq. (4)). The equilibrium condition is that

$$\ddot{r} = 0 \quad \longrightarrow \quad \frac{l_\theta^2 \sin^2 \alpha}{m^2 r_0^3} = g \sin\alpha \cos\alpha \ , \tag{6}$$

and the frequency of small oscillations about the (stable) equilibrium position is found by inserting $r = r_0 + \delta$ (with δ an infinitesimal displacement) in the r equation of motion and expanding linearly in δ and its derivatives, which gives

$$\ddot{\delta} - \frac{l_\theta^2 \sin^2 \alpha}{m^2 (r_0 + \delta)^3} + g \sin\alpha \cos\alpha = \ddot{\delta} - \frac{l_\theta^2 \sin^2 \alpha}{m^2 r_0^3} \left(1 - 3\frac{\delta}{r_0}\right) + g \sin\alpha \cos\alpha = 0 \ , \tag{7}$$

which, using Eq. (6) gives

$$\ddot{\delta} + \frac{3 l_\theta^2 \sin^2 \alpha}{m^2 r_0^4} \delta = 0 \ , \tag{8}$$

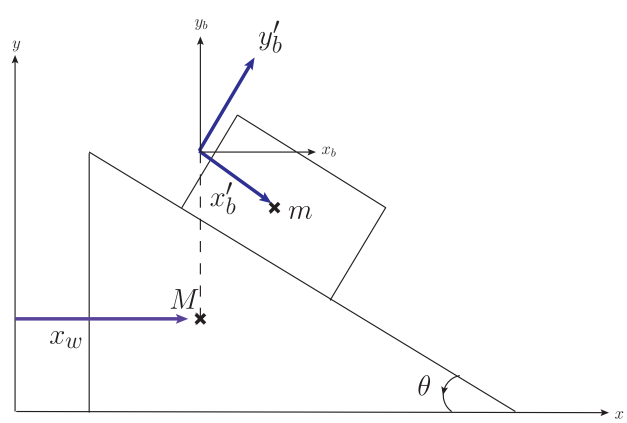
and shows that the bead is performing small oscillations of frequency Ω about $r = r_0$, where

$$\Omega^2 = \frac{3 l_\theta^2 \sin^2 \alpha}{m^2 r_0^4} = \frac{3g}{r_0} \sin\alpha \cos\alpha \ . \tag{9}$$

Problem 2

Since there is no rotational motion, the motion of the wedge and of the block is completely described by the motion of their center(s) of mass. The problem is planar, so we have two coordinates for each center of mass. Let's call them (x_w, y_w) for the center of mass of the wedge and (x_b, y_b) for the center of mass of the block. We also have two constraints:

$$y_w = \text{constant} \quad \text{and} \qquad \frac{y_b}{x_b} = \tan\theta \ , \tag{10}$$



so we will need only two generalized coordinates. We pick x_w for the wedge and x_b' for the block, where x_b' is neither x_b nor y_b but is the coordinate along the incline (i.e. it is the x coordinate of a rotate system of Cartesian coordinates that has the x axis along the incline and the y axis orthogonal to it; we notice that in this coordinate system the second constraint becomes simply y'_b =constant as well). The relation between x_b and y_b and the chosen set of generalized coordinates is

$$\begin{aligned} x_b &= x_w + x'_b \cos\theta + \text{constant} \ , \\ y_b &= -x'_b \sin\theta + \text{constant} \end{aligned} \tag{11}$$

where the constant terms are irrelevant either in the definition of the Lagrangian or in the form of the equations of motion, so we do not specify them any further.

The kinetic energy of the system can then be written as

$$T = \frac{1}{2}M\dot{x}_w^2 + \frac{1}{2}\left(\dot{x}_b^2 + \dot{y}_b^2\right) = \frac{1}{2}M\dot{x}_w^2 + \frac{1}{2}\left(\dot{x}_w^2 + \dot{x}_b'^2 + 2\dot{x}_w\dot{x}'_b\cos\theta\right) \ , \tag{12}$$

while the potential energy is

$$V = mgy_b = -mgx'_b \sin\theta + \text{constant} \ , \tag{13}$$

where the constant term can be dropped because it does not affect the equations of motion. The Lagrangian is therefore

$$L = T - V = \frac{1}{2}M\dot{x}_w^2 + \frac{1}{2}\left(\dot{x}_w^2 + \dot{x}_b'^2 + 2\dot{x}_w\dot{x}'_b\cos\theta\right) + mgx'_b\sin\theta \ . \tag{14}$$

The equation of motion for x_w is

$$\frac{d}{dt}\left[M\dot{x}_w + m(\dot{x}_w + \dot{x}'_b\cos\theta)\right] = 0 \tag{15}$$

which expresses the conservation of the x component of the linear momentum of the system. It provides a relation between $\ddot{x}_w$ and $\ddot{x}'_b$ of the form

$$\ddot{x}_w = \frac{-m\cos\theta}{M+m}\ddot{x}'_b \ . \tag{16}$$

Furthermore, the equation of motion for x'_b is

$$\frac{d}{dt}(\dot{x}_w + \dot{x}'_b\cos\theta) - mg\sin\theta = 0 \ , \tag{17}$$

which gives

$$\ddot{x}'_b = g\sin\theta - \cos\theta\ddot{x}_w \ . \tag{18}$$

Solving the system of Eqs. (16) and (18) we get

$$\begin{aligned} \ddot{x}_w &= -\frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta} \ , \\ \ddot{x}'_b &= \frac{g\sin\theta}{1 - \frac{m\cos^2\theta}{M+m}} \ . \end{aligned} \tag{19}$$

Problem 3 (Goldstein 2.5)

The action is

$$S = \int_0^{t_0} L dt = \int_0^{t_0} \left(\frac{1}{2} m\dot{x}^2 + Fx \right) dt.$$

From the problem statement we know $\dot{x}(t) = B + 2Ct$, and we can set the initial conditions

$$\begin{aligned} x(0) &= A = 0 \Rightarrow A = 0 \\ x(t_0) &= Bt_0 + Ct_0^2 = a \Rightarrow B = \frac{1}{t_0}(a - Ct_0^2). \end{aligned}$$

Now we can explicitly calculate the action:

$$\begin{aligned} S &= \int_0^{t_0} \left[\frac{1}{2} m(B + 2Ct)^2 + F(Bt + CT^2) \right] dt \\ &= \int_0^{t_0} \left[\frac{1}{2} mB^2 + 2mBCt + 2mC^2t^2 + FBt + FCt^2 \right] dt \\ &= \frac{1}{2} mB^2 t_0 + mBCt_0^2 + \frac{2}{3} mC^2 t_0^3 + \frac{1}{2} FBt_0^2 + \frac{1}{3} FCt_0^3 \\ &= \frac{1}{2} mB^2 t_0 + (mBC + \frac{1}{2} FB)t_0^2 + \frac{1}{3}(2mC^2 + FC)t_0^3 \\ &= \frac{1}{2} m \frac{1}{t_0^2}(a - Ct_0^2)^2 t_0 + (mC + \frac{1}{2} F)\frac{1}{t_0}(a - Ct_0^2)t_0^2 + \frac{1}{3}(2mC^2 + FC)t_0^3 \\ &= \frac{1}{2} m \frac{1}{t_0}(a^2 + C^2 t_0^4 - 2aCt_0^2) + t_0(mC + \frac{1}{2} F)(a - Ct_0^2) + \frac{1}{3}(2mC^2 + FC)t_0^3 \\ &= \frac{1}{2} m \frac{a^2}{t_0} + \frac{1}{2} mC^2 t_0^3 - maCt_0 + maCt_0 - mC^2 t_0^3 + \frac{1}{2} aFt_0 - \frac{1}{2} CFt_0^3 + \frac{2}{3} mC^2 t_0^3 + \frac{1}{3} FCt_0^3 \\ &= \frac{1}{2} ma^2 \frac{1}{t_0} + \frac{1}{2} aFt_0 + \frac{1}{6}(mC^2 - FC)t_0^3. \end{aligned}$$

Notice that since we have already set our initial conditions, the only unknown in our equation of motion is C. Thus, this is what we want to find the minimum with respect to, $\partial S / \partial C = 0$.

$$\frac{\partial S}{\partial C} = \frac{1}{6}(2mC - F)t_0^3 = 0.$$

Thus we have

$$\begin{aligned} C &= \frac{F}{2m} \\ B &= \frac{1}{t_0} \left(a - \frac{F}{2m} t_0^2 \right). \end{aligned}$$

Note that we can check this is a minimum by looking at the second derivative;

$$\frac{\partial^2 S}{\partial C^2} = \frac{1}{3} mt_0^3 > 0.$$

Problem 4 (Goldstein 2.22)

In general, the conservation of the total mechanical energy of the system (defined as $E = T + U$) needs to be established by looking explicitly at the total differential of the energy with respect to time, i.e. $\frac{dE}{dt}$. Under special circumstances, i.e. when the relation between cartesian and generalized coordinates does not depend on time and the active forces are conservative, it is true that $\frac{dE}{dt} = -\frac{\partial L}{\partial t}$ and the conservation of energy can be established by simply looking at the explicit time dependence/independence of the Lagrangian.

In the case of the problem, since the equation of the constraint is time-dependent ($\sigma(\mathbf{r}, t) = 0$), the relation between cartesian and generalized coordinates is in general time dependent and the conservation of energy cannot be deduced from the Lagrangian. In general we can then write that,

$$\begin{aligned} \frac{dE}{dt} &= \frac{dT}{dt} + \frac{dU}{dt} \\ &= \frac{\partial T}{\partial q_i}\dot{q}_i + \frac{\partial T}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial T}{\partial t} + \frac{\partial U}{\partial q_i}\dot{q}_i + \frac{\partial U}{\partial t} \ , \end{aligned} \tag{20}$$

where repeated indeces indicate summation. Using that for a system subject to both potential ($Q_i = -\frac{\partial U}{\partial q_i}$) and non-potential ($\tilde{Q}_i$) forces the Euler-Lagrange equations read,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = -\frac{\partial U}{\partial q_i} + \tilde{Q}_i \ , \tag{21}$$

we can recast Eq. (20) in the following form,

$$\frac{dE}{dt} = \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} + \frac{\partial U}{dq_i} - \tilde{Q}_i\right)\dot{q}_i + \frac{\partial T}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial T}{\partial t} + \frac{\partial U}{\partial q_i}\dot{q}_i + \frac{\partial U}{\partial t} \tag{22}$$

$$\begin{aligned} &= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\dot{q}_i\right) + \frac{\partial T}{dt} + 2\frac{\partial U}{\partial q_i}\dot{q}_i + \frac{\partial U}{dt} - \tilde{Q}_i\dot{q}_i \\ &= 2\frac{dT}{dt} - \frac{d(T_1 + 2T_0)}{dt} + \frac{\partial T}{dt} + 2\frac{dU}{dt} - \frac{\partial U}{\partial t} - \tilde{Q}_i\dot{q}_i \ , \end{aligned} \tag{23}$$

or equivalently,

$$\frac{dE}{dt} = \frac{d(T_1 + 2T_0)}{dt} - \frac{\partial T}{\partial t} + \frac{\partial U}{\partial t} + \tilde{Q}_i\dot{q}_i \ , \tag{24}$$

where we have written the kinetic energy T as a polinomial in the $\dot{q}_i$,

$$T = A_{ij}\dot{q}_i\dot{q}_j + B_i\dot{q}_i + C \equiv T_2 + T_1 + T_0 \ , \tag{25}$$

with,

$$A_{ij} = \frac{\partial \mathbf{r}}{\partial q_i}\cdot\frac{\partial \mathbf{r}}{\partial q_j} \ , \quad B_i = \frac{\partial \mathbf{r}}{\partial q_i}\cdot\frac{\partial \mathbf{r}}{\partial t} \ , \quad \text{and} \ C = \frac{\partial \mathbf{r}}{\partial t}\cdot\frac{\partial \mathbf{r}}{\partial t} \ , \tag{26}$$

and we have used that,

$$\frac{dT_2}{dt} = 2T_2 \ , \quad \frac{dT_1}{dt} = T_1 \ , \quad \text{and} \quad \frac{dT_0}{dT} = 0 \ . \tag{27}$$

If the constraints are time dependent (or *reonomic*), the relations between cartesian and generalized coordinates will in general be also time dependent ($\mathbf{r} = \mathbf{r}(q_i, t)$) and one or more terms on the right hand side of Eq. (24) will be non zero causing the total mechanical energy of the system not to be conserved. Physically, we can interpret this by observing that, in the case or reonomic or time-dependent constraints, the work of the forces of constraints is non zero since they do not remain orthogonal to the trajectory during the time evolution of the system (given that the constraints change with time). The work of the forces of constraints per unit time corresponds indeed to the last term in Eq. (24), and as we see it is just one of the reasons why the total mechanical energy of the system is not conserved. Occasionally the explicit time-dependence of either kinetic or potential energies, or both, can contribute to it as well.

2 Non-graded Problems

Problem 5 (Goldstein 2.19)

If the mass distribution has a given symmetry, so will the potential and therefore so will the Lagrangian. From the symmetry, we deduce the conserved quantity.

- (a) The force does not depend on $(x, y) \to (p_x, p_y)$ conserved. It also does not depend on the angle of rotation about $\hat{\mathbf{z}}$, so l_z is conserved as well.
- (b) The force does not depend on x, so p_x is conserved.
- (c) The force does not depend on z so p_z is conserved. It is also independent of the angle of rotating about $\hat{\mathbf{z}}$, so l_z is also conserved.
- (d) The force does not depend on the angle of rotation about $\hat{\mathbf{z}}$ (although it is now a function of z since it is finite), so l_z is conserved while p_z is not.
- (e) The force does not depend on z so p_z is conserved.
- (f) The force does not depend on the angle of rotation about $\hat{\mathbf{z}}$, so l_z is conserved.
- (g) For h the distance between coils, the combination $hp_z + l_z$ is conserved.

Problem 6 (Goldstein 2.6)

First assume the Earth has a uniform mass density ρ, so that the mass distribution is

$$\begin{aligned} M(r) &= 4\pi^2 \int_0^r \rho r^2 dr = \frac{4}{3}\pi\rho r^3, \\ \rho &= \frac{M(R)}{\frac{4}{3}\pi R^3} = \frac{M_E}{\frac{4}{3}\pi R^3} \Rightarrow M(r) = M_E \frac{r^3}{R^3}, \end{aligned}$$

Where R is the radius of the Earth and M_E is the mass. Using this we can find the gravitational force and corresponding potential on a mass m,

$$\begin{aligned} \mathbf{F}(r) &= -\frac{GM(r)m}{r^2}\hat{\mathbf{r}} = -\frac{GM_E m}{R^3} r\hat{\mathbf{r}}, \\ V(r) &= -\int_0^r (\mathbf{F}(r') \cdot \hat{\mathbf{r}'}) dr' = \frac{GM_E m}{2R^3} r^2. \end{aligned}$$

So at a distance r from the center of the Earth we can find the energy to be

$$E = T + V = \frac{1}{2}mv^2 + \frac{GM_E m}{2R^3}r^2.$$

Since the gravitational force is conservative we can use energy conservation to find the velocity. If the velocity at the surface is zero we have

$$E(R) = E(r) \Rightarrow \frac{GM_E m}{2R^3}R^2 = \frac{1}{2}mv^2 + \frac{GM_E m}{2R^3}r^2 \Rightarrow v(r) = \sqrt{\frac{GM_E}{R^3}(R^2 - r^2)}.$$

We want to find the curve that minimizes the time between points a and b,

$$t = \int_a^b \frac{ds}{v(r)}.$$

If we consider coordinates $\{x, y\}$ which define the plane passing through the center of the Earth and the two points at the ends of the curve, we can expressed the arc length and the time traveled as

$$\begin{aligned}
ds &= \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\
&= \sqrt{x'^2 + y'^2}\, d\theta, \\
t &= \int_0^\theta \sqrt{\frac{R^3}{GM_E}\frac{\left[\left(\frac{dx}{d\theta'}\right)^2 + \left(\frac{dy}{d\theta'}\right)^2\right]}{R^2 - r^2}}\, d\theta' \\
&= \int_0^\theta \sqrt{k^2 \frac{x'^2 + y'^2}{R^2 - (x^2 + y^2)}}\, d\theta'.
\end{aligned}$$

We have defined $k = \sqrt{R^3/GM_E}$. To minimize the function

$$f(x, y, x', y') = k\sqrt{\frac{x'^2 + y'^2}{R^2 - (x^2 + y^2)}} \tag{28}$$

we consider the Euler-Lagrange equations:

$$\begin{aligned}
\frac{d}{d\theta}\frac{\partial f}{\partial x'} - \frac{\partial f}{\partial x} &= 0 \\
\frac{d}{d\theta}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} &= 0.
\end{aligned}$$

Since f is not explicitly a function of θ, the Euler-Lagrange equations are equivalent to

$$\begin{aligned}
f - x'\frac{\partial f}{\partial x'} &= \text{const.} = C_1 \\
f - y'\frac{\partial f}{\partial y'} &= \text{const.} = C_2
\end{aligned}$$

Using our function from eq (28) we can write this as:

$$
\begin{aligned}
\frac{y'^2}{x'^2+y'^2}\sqrt{\frac{x'^2+y'^2}{R^2-(x^2+y^2)}} &= \frac{C_1}{k} \\
\frac{x'^2}{x'^2+y'^2}\sqrt{\frac{x'^2+y'^2}{R^2-(x^2+y^2)}} &= \frac{C_2}{k} \\
\Downarrow \quad & \quad \Downarrow \\
\sqrt{\frac{x'^2+y'^2}{R^2-(x^2+y^2)}} &= \frac{C_1}{k}+\frac{C_2}{k} \\
\frac{x'^2+y'^2}{R^2-(x^2+y^2)} &= \frac{C^2}{k^2} \\
x'^2+y'^2 &= \frac{C^2}{k^2}\left[R^2-(x^2+y^2)\right], \qquad (29)
\end{aligned}
$$

where $C = C_1 + C_2$. The solution to this equation is a hypocycloid with parametric equations

$$
\begin{aligned}
x(\theta) &= (R-b)\cos\theta + b\cos\left(\frac{R-b}{b}\theta\right) \\
y(\theta) &= (R-b)\sin\theta - b\sin\left(\frac{R-b}{b}\theta\right).
\end{aligned}
$$

These give the coordinates of a point on a circle of radius b rolling with no slipping inside a circle of radius R. The constant b is determined by finding

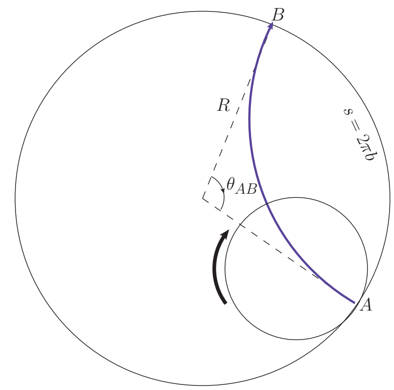
$$
\begin{aligned}
x'(\theta) &= -(R-b)\sin\theta - (R-b)\sin\left(\frac{R-b}{b}\theta\right) \\
y'(\theta) &= (R-b)\cos\theta - (R-b)\cos\left(\frac{R-b}{b}\theta\right).
\end{aligned}
$$

and substituting into eq. (29). We find

$$\frac{C^2}{k^2} = \frac{R-b}{b} \Rightarrow C = k\sqrt{\frac{R-b}{b}}.$$

The time is takes to travel between two points on the Earth's surface is

$$
\begin{aligned}
t_{AB} &= \int_A^B \frac{ds}{v(r)} = \int_A^B k\sqrt{\frac{x'^2+y'^2}{R^2-(x^2+y^2)}}d\theta \\
&= k\frac{C}{k}\int_A^B d\theta = C d\theta_{AB} \\
&= C\frac{2\pi b}{R} = k\sqrt{\frac{R-b}{b}}\frac{2\pi b}{R} \\
&= k\sqrt{\frac{R-\frac{s}{2}\pi}{\frac{s}{2}\pi}}\frac{s}{R} \\
&= k\sqrt{\frac{2\pi R-s}{s}}\frac{s}{R} \\
&= \sqrt{\frac{sR}{GM_E}(2\pi R-s)} \approx 1640\text{s} \approx 27.4\text{min.}
\end{aligned}
$$



We have used the fact that the distance will simply be an arc length of a circle connecting the two points A and B (see figure). We have also used the constants $s = 4800$ km, $R \sim 6371$ km, $M_E \sim 5976 \times 10^{24}$ kg, $G = 6.6726 \times 10^{-11}$ Nm^2/kg^2. At the deepest point the tunnel would be $2b = s/\pi \approx 1528$ km!

Using another property of the hypocycloid we can determine the length of the tunnel:

$$
\begin{aligned}
l(\theta) &= \frac{8(R-b)b}{R}\sin^2\left(\frac{R}{4b}\theta\right) \\
l\left(\frac{2\pi b}{R}\right) &= 8\frac{\left(R-\frac{s}{2\pi}\right)\frac{s}{2\pi}}{R}\sin^2\left(\frac{R}{4b}\frac{2\pi b}{R}\right) \\
&= 8\left(R-\frac{s}{2\pi}\right)\frac{s}{2\pi R} \approx 5378 \text{ km.}
\end{aligned}
$$