PHY 5246: Theoretical Dynamics, Fall 2015

Assignment # 4, Solutions

1 Graded Problems

Problem 1

(1.a)

The problem has spherical symmetry and is therefore naturally solved using spherical coordinates (see figure). The fixed length of the pendulum gives the constraint: $r = l$ and reduces the number of generalized coordinates to two: $\{\theta, \phi\}$.

The kinetic and potential energies of the pendulum are

$$\begin{aligned} T &= \frac{1}{2}m\left(l^2\dot{\theta}^2 + l^2\sin^2\theta\dot{\phi}^2\right) \ , & (1)\\ V &= mgl(1-\cos\theta) \ , & (2) \end{aligned}$$

and the Lagrangian is:

$$L = T - V = \frac{1}{2}m\left(l^2\dot{\theta}^2 + l^2\sin^2\theta\dot{\phi}^2\right) - mgl(1-\cos\theta) \ . \tag{3}$$

Since the Lagrangian does not depend on ϕ, the Euler-Lagrange equation of motion for ϕ states the conservation of the angular momentum with respect to the vertical (or z) axis:

$$\frac{d}{dt}\left(ml^2\sin^2\theta\dot{\phi}\right) = 0 \quad\longrightarrow\quad ml^2\sin^2\theta\dot{\phi} = \text{constant} = p_\phi \quad\longrightarrow\quad \dot{\phi} = \frac{p_\phi}{ml^2\sin^2\theta} \ . \tag{4}$$

On the other hand, the Euler-Lagrange equation of motion for θ reads

$$l^2\ddot{\theta} - l^2\dot{\phi}^2\sin\theta\cos\theta + gl\sin\theta = 0 \ . \tag{5}$$

Substituting in Eq. (5) $\dot{\phi}$ in terms of p_ϕ, as derived in Eq. (4), the equation for θ reduces to a one dimensional equation in the only variable θ (and its derivatives) of the form

$$\ddot{\theta} - \frac{p_\phi^2}{m^2l^4\sin^3\theta}\cos\theta + \frac{g}{l}\sin\theta = 0 \ . \tag{6}$$

(1.b)

The case $\phi = \phi_0$, i.e. $\dot{\phi} = 0$ is trivial and reduced to the plane simple pendulum. Indeed, Eq. (5) reduces to

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \ . \tag{7}$$

(1.c)

The case $\theta = \theta_0$ corresponds to the case of a *conical pendulum*, since the pendulum describes a conical surface during its motion. $\theta = \theta_0$ is an equilibrium point, and therefore needs to verify the condition $\ddot{\theta} = \dot{\theta} = 0$. Imposing this condition in Eq. (5) we get that, in order to realize the condition of *conical pendulum* we need to have $\dot{\phi} = \dot{\phi}_0$ such that

$$\dot{\phi}_0^2 \sin\theta_0 \cos\theta_0 = \frac{g}{l}\sin\theta_0 \quad \longrightarrow \quad \dot{\phi}_0^2 = \frac{g}{l}\frac{1}{\cos\theta_0} = \frac{g}{l}\sec\theta_0 \ . \tag{8}$$

$\theta = \theta_0$ is indeed a stable equilibrium position, as we can prove by studying the motion of the pendulum for small displacements (in θ) from the equilibrium angle $\theta = \theta_0$. In order to do that, let's keep $\dot{\phi} = \dot{\phi}_0$ and let's consider a small displacement from the equilibrium position, i.e. $\theta = \theta_0 + \delta$, for δ infinitesimal. Plugging this into Eq. (5), we get

$$\ddot{\delta} + \frac{g}{l}\sin(\theta_0 + \delta) - \frac{g}{l}\frac{\sin^4\theta_0}{\cos\theta_0}\frac{\cos(\theta_0+\delta)}{\sin^3(\theta_0+\delta)} = 0 \ . \tag{9}$$

Expanding in δ and keeping only up to the linear terms in δ, we find

$$\begin{aligned}
&\ddot{\delta} + \frac{g}{l}\left[\sin\theta_0 + \cos\theta_0\,\delta - \frac{\sin^4\theta_0}{\cos\theta_0}\frac{\cos\theta_0\left(1-\delta\frac{\sin\theta_o}{\cos\theta_0}\right)}{\sin^3\theta_0\left(1+\delta\frac{\cos\theta_0}{\sin\theta_0}\right)^3}\right] = 0 \ , \qquad (10)\\
&\ddot{\delta} + \frac{g}{l}\left[\sin\theta_0 + \cos\theta_0\,\delta - \sin\theta_0\left(1-\delta\frac{\sin\theta_0}{\cos\theta_0} - 3\delta\frac{\cos\theta_0}{\sin\theta_0}\right)\right] = 0 \ , \\
&\ddot{\delta} + \frac{g}{l}\left(3\cos\theta_0 + \sec\theta_0\right)\delta = 0 \ ,
\end{aligned}$$

which is the equation of an harmonic oscillator with frequency

$$\omega^2 = \frac{g}{l}\left(3\cos\theta_0 + \sec\theta_0\right) \ . \tag{11}$$

So, a *conical pendulum* rotates about the vertical axis with constant velocity $\dot{\phi}_0$ (given in Eq. (8)) while it performs small oscillations about the $\theta = \theta_0$ position with frequency ω (given in Eq. (11)). The period of small oscillations in θ is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g(3\cos\theta_0 + \sec\theta_0)}} \ . \tag{12}$$

During one complete oscillation in θ the pendulum sweeps an angle $\phi_1 > \pi$, since

$$\phi_1 = \dot{\phi}_0 T = \sqrt{\frac{g}{l}\sec\theta_0}\, 2\pi\sqrt{\frac{l}{g(3\cos\theta_0 + \sec\theta_0)}} = \frac{2\pi}{\sqrt{1+3\cos^2\theta^0}} > \pi \ . \tag{13}$$

If we imagine to project the motion of the pendulum on the (x, y) plane, it will look like a curve that could close after m turns if the condition $2\pi n = mT$ is verified. On the other hand, if we imagine to trace the motion of the pendulum on a sphere, we would see it oscillating between two circles (one lower and one higher of the $\theta = \theta_0$ circle) describing a sort of sinusoidal curve on the surface of the sphere.

Problem 2

The problem is planar (given the symmetry in the azimuthal angle), and can be treated using polar coordinates $\{r, \theta\}$. Normally, we would impose the constraint $r = a$ and work with just one generalized coordinate, θ. However, the problem asks for the angle at which the particle leaves the hemisphere, which corresponds to the angle at which the force of the constraint vanishes. We then need to determine the force of the constraint, keeping the full set of coordinates, $\{r, \theta\}$, and using the method of Lagrange undetermined multipliers.

We have one constraint, $r - a = 0$, and therefore we introduce one undetermined multiplier, λ. Comparing the equation of the constraint, $r - a = 0$, to the generic constraint equation in differential form

$$a_r dr + a_\theta d\theta = 0 \quad \text{we deduce:} \quad a_r = 1 \ , \ a_\theta = 0 \ .$$

The kinetic and potential energies of the particle are

$$\begin{aligned} T &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \ , \\ V &= mgr\cos\theta \ , \end{aligned} \tag{14}$$

and the Lagrangian is

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta \ . \tag{15}$$

The equation of motions will then be, for r,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = a_r\lambda \quad \longrightarrow \quad m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta = \lambda \ , \tag{16}$$

and for θ,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = a_\theta\lambda \quad \longrightarrow \quad mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mgr\sin\theta = 0 \ . \tag{17}$$

We will solve them for $r(t)$, $\theta(t)$ and λ, together with the constraint equation $r = a$ (i.e. $\dot{r} = \ddot{r} = 0$). After using the equation of the constraint, Eqs. (16) and (17) become:

$$ma\dot{\theta}^2 - mg\cos\theta + \lambda = 0 \tag{18}$$

$$mga\sin\theta - ma^2\ddot{\theta} = 0 \ . \tag{19}$$

Noticing that

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} \ , \tag{20}$$

we can derive, from Eq. (19), that

$$\ddot{\theta} = \frac{g}{a}\sin\theta = \dot{\theta}\frac{d\dot{\theta}}{d\theta} \ , \tag{21}$$

which, upon integration, yields

$$\frac{1}{2}\dot{\theta}^2 = -\frac{g}{a}\cos\theta + \frac{g}{a} \ , \tag{22}$$

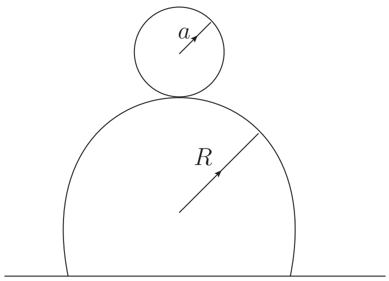
where we have assumed that $\dot{\theta} = 0$ at $t = 0$, when $\theta = 0$. Substituting $\dot{\theta}^2$ into Eq. (18) we get

$$\lambda = mg(3\cos\theta - 2) \ , \tag{23}$$

and therefore the particle leaves the hemisphere when

$$\lambda = 0 \quad \longrightarrow \quad \theta = \arccos\left(\frac{2}{3}\right) \ . \tag{24}$$

Problem 3 (Goldstein 2.14)



There are two constraints in this problem: (1) the distance from the center of the cylinder to the center of the hoop is $R+a$, and (2) there is no slipping, *i.e.* the velocity of the contact point between the cylinder and the hoop is zero. Describe this motion using the coordinates of the center of mass of the hoop and the rotation angle of the hoop about the center of mass. The constraints are thus

$$r = R + a \tag{25}$$

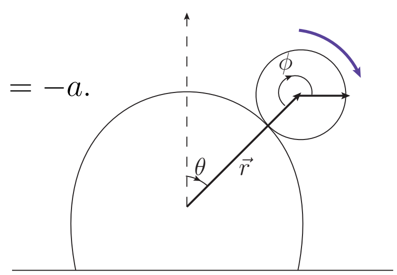
$$(R+a)\dot{\theta} - a\dot{\phi} = 0 \tag{26}$$

We will use the method of Lagrange multipliers to find the normal force on the hoop. The value of θ for which the force is zero will correspond to the point at which the hoop falls of the cylinder. The generalized coordinates will be $\{r, \theta, \phi\}$.

The equations of constraints are

$$r = R + a \to dr = 0 \to a_{1,r} = 1,\ a_{1,\theta} = a_{1,\phi} = 0$$

$$(R+a)\dot{\theta} - a\dot{\phi} = 0 \to (R+a)d\theta - ad\phi = 0 \to a_{2,r} = 0,\ a_{2,\theta} = R+a,\ a_{2,\phi} = -a.$$



The Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}(ma^2)\dot{\phi}^2 - mgr\cos\theta,$$

and the equations of motion (via Euler-Lagrange) are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \lambda_1 a_{1,r} \Rightarrow m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta = \lambda_1,$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \lambda_2 a_{2,\theta} \Rightarrow 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} - mgr\sin\theta = (R+a)\lambda_2,$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = \lambda_2 a_{2,\phi} \Rightarrow ma\ddot{\phi} = -a\lambda_2,$$

where the Lagrange multipliers are λ_1, λ_2. These need to be solved together with the equations of constraint (25) (which implies $\dot{r} = \ddot{r} = 0$) and (26). We find

$$-m(R+a)\dot{\theta}^2 + mg\cos\theta = \lambda_1, \tag{27}$$

$$m(R+a)^2\ddot{\theta} - mg(R+a)\sin\theta = (R+a)\lambda_2, \tag{28}$$

$$ma^2\ddot{\phi} = -a\lambda_2. \tag{29}$$

Note that

- λ_1 corresponds to the r-coordinate and is the normal force on the hoop. Thus, the condition we are looking for is $\lambda_1 = 0$.
- λ_2 corresponds to the rotation of the hoop on the cylinder (in either θ or ϕ), and is thus the tangent friction that make the "no-skipping" condition possible.

Solving the system of equations:

$$(29): \qquad ma^2\frac{R+a}{a}\ddot{\theta} = -a\lambda_2 \to \lambda_2 m(R+a)\ddot{\theta},$$

$$(28): \qquad m(R+a)^2\ddot{\theta} - mg(R+a)\sin\theta = -m(R+a)^2\ddot{\theta} \to 2(R+a)\ddot{\theta} - g\sin\theta = 0$$

$$\Rightarrow \ddot{\theta} = \frac{g\sin\theta}{2(R+a)}.$$

Multiply this last equation by $\dot{\theta}$ and rewrite it as

$$\frac{d}{dt}\left(\frac{1}{2}\dot{\theta}^2\right) = \frac{g}{2(R+a)}\sin\theta\dot{\theta} = \frac{g}{2(R+a)}\frac{d}{dt}(\cos\theta)$$

$$\Rightarrow \frac{1}{2}\dot{\theta}^2 = \frac{g}{2(R+a)}\cos\theta + \text{const.}$$

Setting $\theta = 0$ and $\dot{\theta} = 0$ as the initial conditions we find that constant term is

$$\text{const.} = -\frac{g}{2(R+a)}$$

which gives us the equation of motion

$$\dot{\theta}^2 = \frac{g}{R+a}(\cos\theta - 1).$$

Now plug this into (27):

$$mg(\cos\theta - 1) + mg\cos\theta = \lambda_1.$$

This gives us the Lagrange multiplier (and force of constraint)

$$\lambda_1 = mg(2\cos\theta - 1),$$

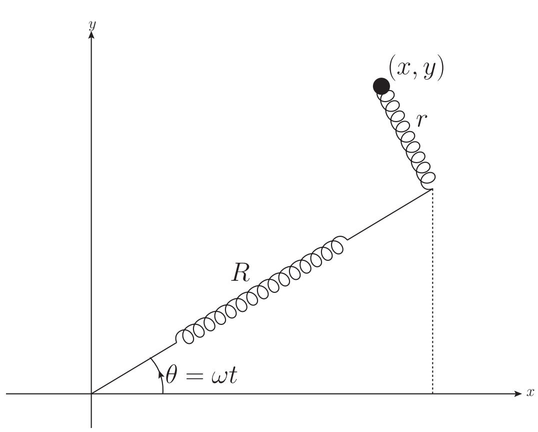
and setting this to zero we can find the angle at which the constraint is violated:

$$\lambda_1 = 0 \to \cos\theta = \frac{1}{2} \to \theta = \frac{\pi}{3} = 60°.$$

Problem 4 (Goldstein 2.21)

(a)+(b)

The laboratory frame (x, y) and rotating frame (R, r) coordinates are defined as,



$$\begin{cases} x = R\cos\theta - r\sin\theta \\ y = R\sin\theta + r\cos\theta \end{cases} \tag{30}$$

$$\Downarrow$$

$$\begin{cases} R = x\cos\theta + y\sin\theta \\ r = -x\sin\theta + y\cos\theta \end{cases} \tag{31}$$

(32)

with $\theta = \omega t$.

Using lab. frame coordinates (x, y), we can write the kinetic and potential energies as,

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)\,, & (33) \\ V &= \frac{1}{2}kr^2 + \frac{1}{2}K\,(R - R_0)^2 = \frac{1}{2}k\,(-x\sin(\omega t) + y\cos(\omega t))^2 + \frac{1}{2}K\,(x\cos(\omega t) + y\sin(\omega t) - R_0)^2\,, \end{aligned}$$

such that the Lagrangian is,

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k\,(-x\sin(\omega t) + y\cos(\omega t))^2 - \frac{1}{2}K\,(x\cos(\omega t) + y\sin(\omega t) - R_0)^2\,, \quad (34)$$

The energy function or Jacobi integral is,

$$h = \frac{\partial L}{\partial \dot{x}}\dot{x} + \frac{\partial L}{\dot{y}}\dot{y} - L = 2T - L = T + V = E\,, \quad (35)$$

as expected since the potential energy does not depend on the velocities and the kinetic energy is a homogeneous function of second degree in the velocities. We can then use that,

$$\frac{dh}{dt} = -\frac{\partial L}{\partial t}\ , \quad (36)$$

to prove that the energy of the system is not conserved, since,

$$\frac{dE}{dt} = \frac{dh}{dt} = -\frac{\partial L}{\partial t} \neq 0\ , \quad (37)$$

given the explicit time dependence in the Lagrangian (via $\theta = \omega t$). This was expected since, in order to keep the steady rotational motion of the system some work is done on the system and this goes into the balance of the mechanical energy of the system, which, by itself (i.e. without including the work done on the system) is not conserved. Stated differently, the mechanical system of the two springs is not isolated.

For completeness, we will also derive here the equations of motion in the lab. frame coordinates:

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \quad &\longrightarrow \quad m\ddot{x} = k(y\cos\theta - x\sin\theta)\sin\theta - K(x\cos\theta + y\sin\theta - R_0)\cos\theta & (38) \\ & \quad m\ddot{x} = kr\sin(\omega t) - K(R - R_0)\cos(\omega t) \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0 \quad &\longrightarrow \quad m\ddot{y} = k(-y\cos\theta + x\sin\theta)\cos\theta - K(x\cos\theta + y\sin\theta - R_0)\sin\theta \\ & \quad m\ddot{y} = -kr\cos(\omega t) - K(R - R_0)\sin(\omega t) \end{aligned}$$

(c)

In the (non-inertial) system of the rotating carriage, the two springs oscillate in the x and y direction respectively, and we chose to call the corresponding coordinates R and r. The kinetic energy is them simply given by,

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\dot{R}^2\ . \quad (39)$$

The force acting on m in the rotating frame ($\vec{\mathbf{F}}_{\mathbf{rot}}$) is,

$$\vec{\mathbf{F}}_{\mathbf{rot}} = \vec{\mathbf{F}}_{\mathbf{lab}} - m\vec{\omega} \times (\vec{\omega} \times \vec{\mathbf{r}}) - 2m\vec{\omega} \times \vec{\mathbf{v}}\,, \quad (40)$$

where $\vec{\mathbf{F}}_{\mathbf{lab}}$ is the force acting on m in the laboratory frame, $\vec{\omega}$ is the vector angular velocity (orthogonal to the plane of motion), while $\vec{\mathbf{r}} = (R, r)$ and $\vec{\mathbf{v}} = (\dot{R}, \dot{r})$ are the position and velocity vectors of m in the rotating frame. The last two terms in Eq. (40) are often referred to as *centrifugal* and *Coriolis* forces. As explained in Sec. 1.5 of Goldstein's book, for forces Q_j that also contain terms dependent on the velocities, a generalized potential function $U(q_j, \dot{q}_j)$ can be defined such that it satisfies,

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j. \tag{41}$$

In our case, it is easy to show that the generalized potential function is of the form,

$$U = \frac{1}{2}kr^2 + \frac{1}{2}K(R - R_0)^2 - \frac{1}{2}m\omega^2(r^2 + R^2) - m\omega(R\dot{r} - r\dot{R})\,. \tag{42}$$

In the rotating system the Lagrangian is then,

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\dot{R}^2 - \frac{1}{2}kr^2 - \frac{1}{2}K(R - R_0)^2 + \frac{1}{2}m\omega^2(r^2 + R^2) + m\omega(R\dot{r} - r\dot{R})\,. \tag{43}$$

From this Lagrangian one can derive the equation of motions corresponding to the (R, r) system of coordinates, i.e.

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{R}} - \frac{\partial L}{\partial R} = 0 \quad &\longrightarrow \quad m\ddot{R} = -K(R - R_0) + m\omega^2 R + 2m\omega\dot{r}\,, \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0 \quad &\longrightarrow \quad m\ddot{r} = -kr + m\omega^2 R - 2m\omega\dot{R}\,,
\end{aligned} \tag{44}$$

and verify that they indeed corresponds to the equations of motions generated by the non-inertial force in Eq. (40). Moreover one can verify that they correspond to the equation of motion in the laboratory frame, when the equations in Eq. (38) are transformed to the rotating frame (using the formalism explained in the *Note* at the end of this problem).

Finally, in the rotating frame the Jacobi integral is,

$$\begin{aligned}
h &= \frac{\partial L}{\partial \dot{R}}\dot{R} + \frac{\partial L}{\dot{r}}\dot{r} - L \\
&= m(\dot{r} + R\omega)\dot{r} + m(\dot{R} - r\omega)\dot{R} - L \\
&= E - mR\omega(\dot{r} + R\omega) + mr\omega(\dot{R} - r\omega)\,.
\end{aligned} \tag{45}$$

Since L is independent of time,

$$\frac{dh}{dt} = -\frac{\partial L}{\partial t} = 0\,, \tag{46}$$

but since T is not a homogeneous function of the velocities (of second degree) then $h \neq E$ and we cannot infer anything from the time (in)dependence of h. Indeed, as seen in **(a)+(b)**, the energy is not conserved and this holds always, independently of the choice of frame or coordinates. It is possible to check this explicitly (this is not needed but it can be an extra check of the consistency of the formalism derived in this part of the problem),

$$\begin{aligned}
\frac{dE}{dt} &= \frac{dh}{dt} + \frac{d}{dt}\Big[mR\omega(\dot{r} + R\omega) - mr\omega(\dot{R} - r\omega)\Big] \\
&= \frac{d}{dt}\Big[m\omega(R\dot{r} - \dot{R}r) + m\omega^2(r^2 + R^2)\Big] \\
&= m\omega(R\ddot{r} - r\ddot{R}) + 2m\omega^2(R\dot{R} + r\dot{r}).
\end{aligned} \tag{47}$$

This expression can be simplified further by using the two equations of motion:

$$\frac{d}{dt}\left[m(\dot{R}-r\omega)\right] m(\dot{r}+R\omega)\omega + K(R-R_0) = 0$$
$$\Rightarrow m\ddot{R} = 2m\dot{r}\omega + m\omega^2 R - K(R-R_0)$$
$$\frac{d}{dt}\left[m(\dot{r}+R\omega)\right] + m(\dot{R}-r\omega)\omega + kr = 0$$
$$\Rightarrow m\ddot{r} = -2m\omega\dot{R} + m\omega^2 r - kr\,. \tag{48}$$

Plugging these two equations into (47) we find that many things cancel and we are left with

$$\frac{dE}{dt} = -k\omega rR - K\omega r(R-R_0) \neq 0\,. \tag{49}$$

<u>Note</u>

The laboratory-frame and rotating-frame coordinates are related by a time-dependent rotation ($\theta = \omega t$), i.e.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = M \begin{pmatrix} R \\ r \end{pmatrix} \longrightarrow \begin{pmatrix} R \\ r \end{pmatrix} = M^t \begin{pmatrix} x \\ y \end{pmatrix} \tag{50}$$

and consequently,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \dot{M}\begin{pmatrix} R \\ r \end{pmatrix} + M\begin{pmatrix} \dot{R} \\ \dot{r} \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} + M\begin{pmatrix} \dot{R} \\ \dot{r} \end{pmatrix} \quad , \tag{51}$$

where,

$$A = \dot{M}M^t = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \quad . \tag{52}$$

In the same way we can show that,

$$\begin{aligned}
\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} &= \dot{A}\begin{pmatrix} x \\ y \end{pmatrix} + A\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} + \dot{M}\begin{pmatrix} \dot{R} \\ \dot{r} \end{pmatrix} + M\begin{pmatrix} \ddot{R} \\ \ddot{r} \end{pmatrix} && (53)\\
&= \dot{A}\begin{pmatrix} x \\ y \end{pmatrix} + A^2\begin{pmatrix} x \\ y \end{pmatrix} + A\,M\begin{pmatrix} \dot{R} \\ \dot{r} \end{pmatrix} + \dot{M}\begin{pmatrix} \dot{R} \\ \dot{r} \end{pmatrix} + M\begin{pmatrix} \ddot{R} \\ \ddot{r} \end{pmatrix} && (54)\\
&= \dot{A}\begin{pmatrix} x \\ y \end{pmatrix} + A^2 M\begin{pmatrix} R \\ r \end{pmatrix} + 2\dot{M}\begin{pmatrix} \dot{R} \\ \dot{r} \end{pmatrix} + M\begin{pmatrix} \ddot{R} \\ \ddot{r} \end{pmatrix} && (55)\\
&= M\left\{A^2\begin{pmatrix} R \\ r \end{pmatrix} + 2A\begin{pmatrix} \dot{R} \\ \dot{r} \end{pmatrix} + \begin{pmatrix} \ddot{R} \\ \ddot{r} \end{pmatrix}\right\} \quad , && (56)
\end{aligned}$$

where we have used that $\dot{A} = 0$, $A = M^t AM$, $AM = MA$, and $A^2M = AMA = MA^2$. From the previous relation we deduce that,

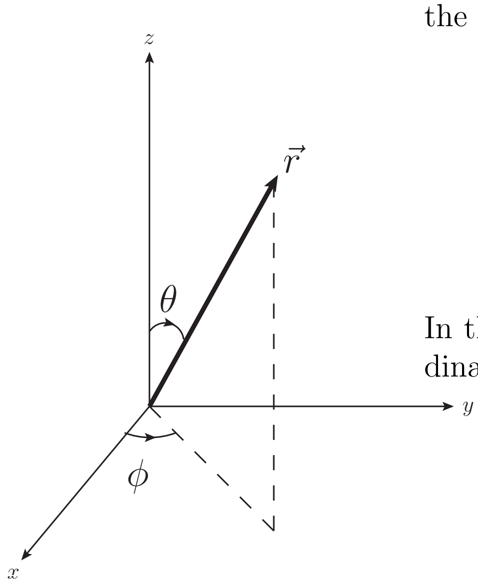
$$M^t\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = A^2\begin{pmatrix} R \\ r \end{pmatrix} + 2A\begin{pmatrix} \dot{R} \\ \dot{r} \end{pmatrix} + \begin{pmatrix} \ddot{R} \\ \ddot{r} \end{pmatrix} = \begin{pmatrix} -\omega^2 R - 2\omega\dot{r} + \ddot{R} \\ -\omega^2 r + 2\omega\dot{R} + \ddot{r} \end{pmatrix} \quad , \tag{57}$$

and therefore we show that the equations of motion in the laboratory frame (see Eq. (38)) transform into the equations of motion on the rotating frame (see Eq. (44)), since,

$$m\,M^t\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -K(R-R_0) \\ -kr \end{pmatrix} = m\begin{pmatrix} -\omega^2 R - 2\omega\dot{r} + \ddot{R} \\ -\omega^2 r + 2\omega\dot{R} + \ddot{r} \end{pmatrix} \quad . \tag{58}$$

2 Non-graded Problems

Problem 5 (Goldstein 2.2)



Consider a rotation about an arbitrary axis that we identify with $\hat{\mathbf{z}}$. This corresponds to the rotation angle ϕ. Let us first consider the one-particle case; if $V(\dot{x}, \dot{y}, \dot{z})$ then

$$\begin{aligned}
\frac{\partial V}{\partial \dot{\phi}} &= \frac{\partial V}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial \dot{\phi}} + \frac{\partial V}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \dot{\phi}} + \frac{\partial V}{\partial \dot{z}}\frac{\partial \dot{z}}{\partial \dot{\phi}} \\
&= \frac{\partial V}{\partial \dot{x}}(-r\sin\theta\sin\phi) + \frac{\partial V}{\partial \dot{y}}(r\sin\theta\sin\phi) \\
&= (\vec{\mathbf{r}} \times \vec{\nabla}_v V(\dot{\vec{\mathbf{r}}})) \cdot \hat{\mathbf{z}} = \vec{\mathbf{n}} \cdot (\vec{\mathbf{r}} \times \vec{\nabla}_v V(\vec{\mathbf{r}}, \dot{\vec{\mathbf{r}}}).
\end{aligned}$$

In the above we have defined $\vec{\mathbf{n}} := \hat{\mathbf{z}}$, and we have used the coordinates

$$\begin{cases} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{cases}$$

$$\begin{cases} \dot{x} &= \dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi + r\dot{\phi}\sin\theta\sin\phi \\ \dot{y} &= \dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi \\ \dot{z} &= \dot{r}\cos\theta - r\dot{\phi}\sin\phi. \end{cases}$$

Therefore we find

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial T}{\partial \dot{\phi}} - \frac{\partial V}{\partial \dot{\phi}} = l_\phi - \sum_i \vec{\mathbf{n}} \cdot (\vec{\mathbf{r}}_i \times \vec{\nabla}_{v_i} V).$$

The new term $\partial V / \partial \dot{\phi}$ only exists for $V = V(\dot{\vec{\mathbf{r}}})$, and we have also extended the result to include many particles. For electromagnetic forces the potential is

$$V = q\Phi - \frac{q}{c}\vec{\mathbf{A}} \cdot \vec{\mathbf{v}}$$

so the gradient is

$$\vec{\nabla}_v V = -\frac{q}{c}\vec{\mathbf{A}}.$$

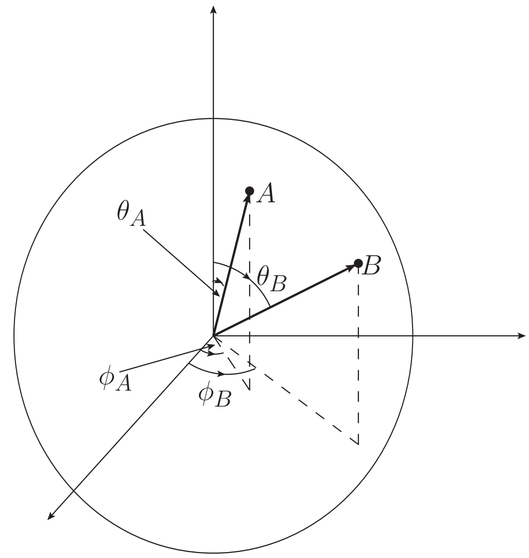
Plugging this in gives us the final result;

$$p_\phi = l_\phi - \sum_i \vec{\mathbf{n}} \cdot \left(\vec{\mathbf{r}}_i \times \frac{q_i}{c}\vec{\mathbf{A}}\right).$$

Problem 6 (Goldstein 2.4)

On a spheroidal surface of radius R:

$$\begin{cases} x &= R\sin\theta\cos\phi \\ y &= R\sin\theta\sin\phi \\ z &= R\cos\theta \end{cases}$$



The arc length is given by

$$ds^2 = R^2 d\theta^2 + R^2\sin^2\theta d\phi^2 \Rightarrow ds = R\sqrt{d\theta^2 + \sin^2 d\phi^2}.$$

Thus the distance between two points (A,B) is given by

$$\begin{aligned} l_{AB} &= \int_A^B ds = R\int_A^B \sqrt{d\theta^2 + \sin^2\theta d\phi^2} \\ &= R\int_{\phi_A}^{\phi_B} d\phi\sqrt{\sin^2\theta + \left(\frac{d\theta}{d\phi}\right)^2}. \end{aligned}$$

Let us define the integrand as

$$f(\theta, \theta') = \sqrt{\sin^2\theta + \theta'^2}.$$

The variational principle tells us that the minimal distance is obtained when θ, θ' satisfy the equation

$$\begin{gathered} \frac{d}{d\theta}\frac{\partial f}{\partial\theta'} - \frac{\partial f}{\partial\theta} = 0 \\ \frac{d}{d\theta}\left(\frac{\theta'}{\sqrt{\sin^2\theta + \theta'^2}}\right) - \frac{\sin\theta\cos\theta}{\sqrt{\sin^2\theta + \theta'^2}} = 0 \\ \frac{\theta''}{f} - \frac{\theta'}{f^2}\frac{\sin\theta\cos\theta\theta' + \theta'\theta''}{f} - \frac{\sin\theta\cos\theta}{f} = 0. \\ (\theta'' - \sin\theta\cos\theta^2_f - \theta'^2(\sin\theta\cos\theta + \theta'') = 0 \\ (\theta'' - \sin\theta\cos\theta)(\theta'^2 + \sin^2\theta) - \theta'^2(\sin\theta\cos\theta + \theta'') = 0 \\ \theta''\sin^2\theta - 2\theta'^2\sin\theta\cos\theta - \sin^3\theta\cos\theta = 0 \\ \sin\theta\left[\theta''\sin\theta - 2\theta'^2\cos\theta - \sin^2\theta\cos\theta\right] = 0 \\ \Rightarrow \theta''\sin^2\theta - 2\theta'^2\cos\theta - \sin^2\theta\cos\theta = 0. \end{gathered} \tag{59}$$

Now let's work a little backwards. Great circles (which is what we are trying to prove being the shortest path between two points on a sphere) are the intersections between a sphere and a plane going through these two points. If

$$\hat{\mathbf{n}} = n_x\hat{\mathbf{x}} + n_y\hat{\mathbf{y}} + n_z\hat{\mathbf{z}}$$

is the vector perpendicular to the plane, the points in the great circle are those such that

$$\hat{\mathbf{n}} \cdot \vec{\mathbf{r}} = 0$$

$$n_x R\cos\theta\cos\phi + n_y R\sin\theta\cos\phi + n_z R\cos\theta = 0$$
$$R\left[\sin\theta(n_x\cos\phi + n_y\sin\phi) + n_z\cos\theta\right] = 0$$

Thus we find the condition satisfied for points on a circle is

$$\frac{\cos\theta}{\sin\theta} = A\cos\phi + B\sin\phi.$$

Now define a function

$$g(\phi) := \frac{\cos\theta(\phi)}{\sin\theta(\phi)}.$$

Find the second derivative of this quantity:

$$\begin{aligned}
\frac{d^2g}{d\phi^2} &= \frac{d}{d\phi}\left(\frac{dg}{d\theta}\frac{d\theta}{d\phi}\right) = \frac{d}{d\theta}\left(\frac{-\sin\theta-\cos^2\theta}{\sin^2\theta}\theta'\right) \\
&= \frac{d}{d\phi}\left(-\frac{1}{\sin^2\theta}\theta'\right) \\
&= \frac{2\cos\theta}{\sin^2\theta}\theta'^2 - \frac{1}{\sin^2\theta}\theta'' \\
&= -\frac{1}{\sin^2\theta}\theta'' + \frac{2\cos\theta}{\sin^2\theta}\theta'^2 \\
&= -\frac{1}{\sin^3\theta}\left[\theta''\sin\theta + 2\cos\theta\theta'^2\right].
\end{aligned}$$

Our minimization condition (59) tell us that the quantity in brackets must be equal to $\sin^2\theta\cos\theta$, so we have

$$\frac{d^2g}{d\phi^2} = -\frac{1}{\sin^3\theta}\sin^2\theta\cos\theta = -\frac{\cos\theta}{\sin\theta} = -g \Rightarrow \frac{d^2g}{d\phi^2} + g = 0$$
$$\Rightarrow g(\phi) = A\cos\phi + B\sin\phi.$$

This is exactly the equation for great circles on a sphere.