PHY 5246: Theoretical Dynamics, Fall 2015

Assignment # 5, Solutions

1 Graded Problems

Problem 1

(1.a)

Using the equation of the orbit or force law

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=-\frac{mr^2}{l^2}F(r) \ , \tag{1}$$

with $r(\theta)=ke^{\alpha\theta}$ one finds

$$\frac{\alpha^2}{r}+\frac{1}{r}=-\frac{mr^2}{l^2}F(r) \ , \tag{2}$$

from which

$$F(r)=-\frac{(1+\alpha^2)l^2}{m}\frac{1}{r^3} \ . \tag{3}$$

(1.b)

For a central force motion we have that

$$\dot{\theta}=\frac{l}{mr^2}=\frac{l}{mk^2}e^{-2\alpha\theta} \ , \tag{4}$$

where l is the magnitude of the conserved angular momentum. We can easily integrate this equation by separation of variables, i.e.

$$e^{2\alpha\theta}d\theta=\frac{l}{mk^2}dt \ \longrightarrow \ \frac{1}{2\alpha}e^{2\alpha\theta}+C=\frac{l}{mk^2}t \ , \tag{5}$$

where C a constant of integration. Isolating the exponential term and taking the logarithm of both l.h.s and r.h.s. one gets

$$\theta(t)-\theta_0=\frac{1}{2\alpha}\ln\left[\frac{2\alpha l}{mk^2}t+C'\right] \ , \tag{6}$$

where $C'=-2\alpha C$ is determined by the initial conditions on θ_0.

Substituting $\theta(t)$ into the expression of $r(\theta)$ one gets

$$r(t)=K\left[\frac{2\alpha l}{mk^2}t+C'\right]^{1/2}=\left[\frac{2\alpha l}{m}t+k^2C'\right]^{1/2} \ , \tag{7}$$

where $K=ke^{\alpha\theta_0}$.

(1.c)

The total energy of the orbit is

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \ , \tag{8}$$

where, using Eqs. (6)-(7), we can calculate the kinetic energy as

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{(1+\alpha^2)l^2}{2m}\frac{1}{r^2} \ , \tag{9}$$

while the potential energy (modulus a constant of integration) is

$$V(r) = -\int F(r)dr = -\int \left[-\frac{(1+\alpha^2)l^2}{m}\frac{1}{r^3}\right] = -\frac{(1+\alpha^2)l^2}{2m}\frac{1}{r^2} \ , \tag{10}$$

and $E = T + V = 0$.

Problem 2

(2.a)

The problem is easily discussed in terms of the effective potential

$$V'(r) = \frac{1}{2}\frac{l^2}{mr^2} + V(r) = \frac{1}{2}\frac{l^2}{mr^2} + \beta r^k \ . \tag{11}$$

In order for a circular orbit to exist the effective potential has to have a minimum for some finite value of r. The minimum condition is

$$\frac{\partial V'(r)}{\partial r} = 0 \quad \longrightarrow \quad -\frac{l^2}{mr^3} + \beta k r^{k-1} = 0 \ , \tag{12}$$

which admits a real solution only if β and k are either both positive or both negative. In which case the radius of the circular orbit is

$$r_0 = \left(\frac{l^2}{mk\beta}\right)^{\frac{1}{k+2}} \ . \tag{13}$$

(2.b)

Since about the equilibrium position $r = r_0$ the system behaves as a linear harmonic oscillator subject to a restoring force $F(r) = -\alpha(r - r_0)$, with potential energy $V'(r) = V'(r_0) + \frac{1}{2}\alpha(r - r_0)^2$, we can find α by simply expanding $V'(r)$ about $r = r_0$ and taking the coefficient of the quadratic term in the expansion. The frequency of small oscillations will then be $\omega_r = (\alpha/m)^{1/2}$ (where the index r indicates that the oscillation are in the radial direction). The expansion of the potential is

$$V'(r) = V'(r_0) + \frac{1}{2}\frac{\partial^2 V'(r)}{\partial r^2}\bigg|_{r=r_0} (r - r_0)^2 + O((r - r_0)^3) \ , \tag{14}$$

such that

$$
\begin{aligned}
\alpha &= \left.\frac{\partial^2 V'(r)}{\partial r^2}\right|_{r=r_0} \\
&= \frac{3l^2}{m}\frac{1}{r_0^4} + \beta k(k-1) r_0^{k-2} \\
&= \left(\frac{l^2}{m\beta k}\right)^{-\frac{4}{k+2}} \left[\frac{3l^2}{m} + \beta k(k-1)\left(\frac{l^2}{m\beta k}\right)^{k+2}\right] \\
&= r_0^{-4}\frac{l^2}{m}(k+2) \ ,
\end{aligned}
\tag{15}
$$

and the frequency of small oscillations ω_r is

$$
\omega_r = \left(\frac{\alpha}{m}\right)^{1/2} = \frac{l}{mr_0^2}\sqrt{k+2} \ . \tag{16}
$$

(2.c)

The ratio of the frequency of small (radial) oscillation, ω_r, to the frequency $\omega_\theta = \dot{\theta}$ of the (nearly) circular motion is

$$
\frac{\omega_r}{\omega_\theta} = \frac{\frac{l}{mr_0^2}\sqrt{k+2}}{\frac{l}{mr_0^2}} = \sqrt{k+2} \ . \tag{17}
$$

The four given cases are:

$$
\begin{aligned}
k = -1 &\longrightarrow \frac{\omega_r}{\omega_\theta} = 1 \\
k = 2 &\longrightarrow \frac{\omega_r}{\omega_\theta} = 2 \\
k = 7 &\longrightarrow \frac{\omega_r}{\omega_\theta} = 3 \\
k = -\frac{7}{4} &\longrightarrow \frac{\omega_r}{\omega_\theta} = \frac{1}{2}
\end{aligned}
\tag{18}
$$

which correspond to r making 1,2,3, or respectively $\frac{1}{2}$ oscillation(s) for each complete revolution in θ.

Problem 3 (Goldstein 3.11)

The reduced system also moves in a circular orbit with some radius $r = a$ (and therefore $\ddot{r} = 0$). The corresponding equation of motion is

$$
\ddot{r} = 0 = \frac{l^2}{ma^3} - \frac{k}{a^2}.
$$

We solve this, using $l = mr^2\dot{\theta}$:

$$
\frac{l^2}{ma^3} = \frac{k}{a^2} \Rightarrow \dot{\theta}^2 = \frac{k}{ma^3}.
$$

$$\dot{\theta} = \omega = \sqrt{\frac{k}{ma^3}} = \frac{2\pi}{\tau} \Rightarrow \tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{ma^3}{k}}. \tag{19}$$

Here we can note that ω is constant and τ must be the period of both the reduced system and the original circular motion.

When the two masses are stopped and then released from rest, they have zero angular momentum $l = 0$, so they just satisfy a radial motion equation of the form

$$-\frac{k}{a} = \frac{1}{2}m\dot{r}^2 - \frac{k}{r},$$

which is easily found using conservation of energy. Therefore

$$\begin{aligned}
\dot{r}^2 &= \frac{2k}{m}\left(\frac{1}{r} - \frac{1}{a}\right) \\
\dot{r} &= -\sqrt{\frac{2k}{m}}\left(\frac{1}{r} - \frac{1}{a}\right)^{1/2} = -\sqrt{\frac{2k}{m}}\left(\frac{a-r}{ra}\right)^{1/2}.
\end{aligned}$$

Integrating the previous relation between $t = 0$ and t, we get,

$$-\int_a^0 \frac{dr}{\sqrt{\frac{a-r}{ra}}} = \sqrt{\frac{2k}{m}}t, \tag{20}$$

where t is the time it takes for the two masses to move from $r = a$ to $r = 0$. Performing this integration:

$$\begin{aligned}
\int_a^0 \frac{dr}{\sqrt{\frac{a-r}{ar}}} &= \sqrt{a}\int_a^0 dr\sqrt{\frac{r}{a-r}} = 2\sqrt{a}\int_{\sqrt{a}}^0 dx\frac{x^2}{\sqrt{a-x^2}} \\
&= 2\sqrt{a}\left[-\frac{x}{2}\sqrt{a-x^2} + \frac{a}{2}\sin^{-1}\left(\frac{x}{\sqrt{a}}\right)\right]_{\sqrt{a}}^0 = -a\frac{\pi}{2}\sqrt{a}.
\end{aligned}$$

In the first line we have changed integration variables with $r = x^2$, and to get to the second line we have used a standard integration table. Thus, from (20) we have

$$a\frac{\pi}{2}\sqrt{a} = \sqrt{\frac{2k}{m}}t \Rightarrow t = \sqrt{\frac{m}{2k}}a\sqrt{a}\frac{\pi}{2}$$

$$t^2 = \frac{m}{2k}a^3\frac{\pi^2}{4} = \frac{\tau^2}{32} \Rightarrow t = \frac{\tau}{4\sqrt{2}}.$$

To get this final result we have used the period we found in (19).

2 Non-graded Problems

Problem 4 (Goldstein 3.19)

(Note that the Yukawa potential is a kind of screened Coulomb potential, and can be used to describe some common particle interactions - pion exchange between nucleons, for instance.)

The force corresponding to the Yukawa potential (for $k, a > 0$) is

$$F(r) = -\frac{k}{r^2}e^{-r/a}.$$

(4.a)

The Lagrangian corresponding to a particle in the Yukawa potential is

$$L = \frac{1}{2}m(\dot{r}^2 r^2\dot{\theta}^2) - V(r) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}e^{-r/a}.$$

The equation of motion for θ simply gives us conservation of angular momentum:

$$mr^2\dot{\theta} = \text{constant} := l.$$

The equation of motion for r is

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{k}{r^2}e^{-r/a} + a\frac{k}{r}e^{-r/a} = 0$$

$$m\ddot{r} - \frac{l^2}{mr^3} + \left(\frac{k}{r^2} + \frac{ak}{r}\right)e^{-r/a} = 0.$$

Using this we can write the energy as:

$$\begin{aligned} E &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \\ &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{l^2}{mr^2} - \frac{k}{r}e^{-r/a} = \frac{1}{2}m\dot{r}^2 + V'(r), \end{aligned}$$

where $V'(r)$ is the effective potential (see figure). Asymptotically, this potential has the feature that for both large ($r \to \infty$) and small ($r \to 0$) it is dominated by the $1/r^2$ term. In the middle regions it will depend on the value of l.

(4.b)

The circular orbit condition is verified (for those values of l when $V'(r)$ has a minimum) if:

$$\frac{\partial V'(r)}{\partial r} = 0 \Rightarrow -\frac{l^2}{mr^2} + \left(\frac{k}{r^2} + \frac{k}{ra}\right)e^{-r/a} = 0$$

$$\frac{l^2}{mk} = r_0 e^{-r_0/a}\left(a + \frac{r_0}{a}\right). \tag{21}$$

In this case we explain what happens when we examine small deviations from $r = r_0$. Take

$$r(\theta) = r_0\left[1 + \delta(\theta)\right]$$

and insert this into the equation for the orbit

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{mr^2}{l^2}F(r) = \frac{mk}{l^2}e^{-r/a}.$$

Using the standard change of variables

$$u := \frac{1}{r} = \frac{1}{r_0}(1 - \delta),$$

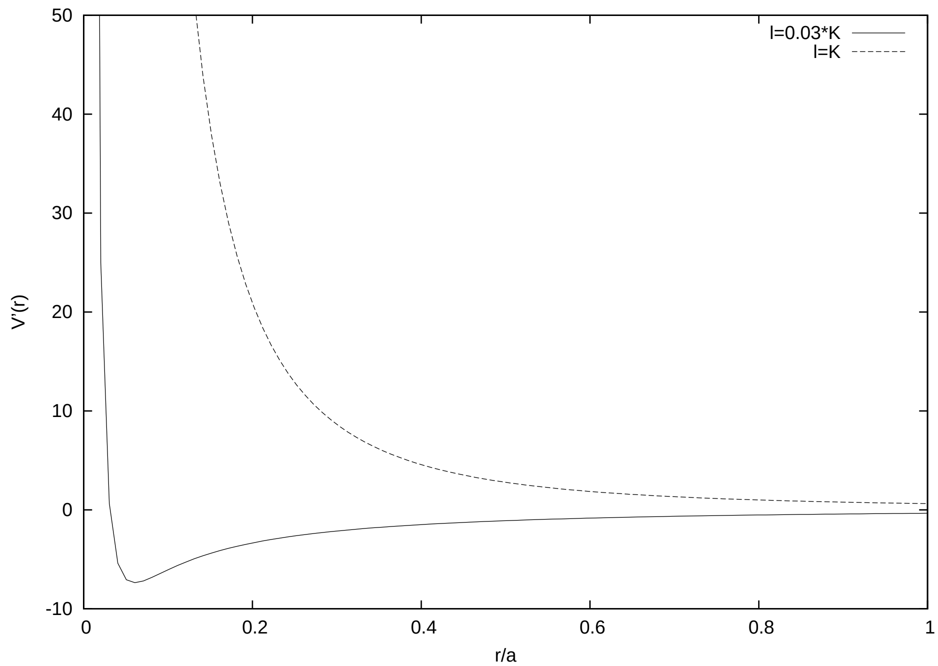

Figure 1: A graph of the effective Yukawa potential for two different vales of angular momentum. Here we have set $k/a = 1$ and $K = \sqrt{2mk}$.

we find that

$$
\begin{aligned}
\frac{d^2u}{d\theta^2} + u &= \frac{mk}{l^2} e^{-1/au} \\
&\Downarrow \\
\frac{d^2u}{d\theta^2} + (1-\delta) &= \frac{mk}{l^2} r_0 e^{-\frac{r_0}{a}(1+\delta)} \left(a - \frac{r_0}{a}\delta\right) \\
&\Downarrow \\
\frac{d^2\delta}{d\theta^2} + \left(1 - \frac{mk}{l^2 a} r_0^2 e^{-r_0/a}\right)\delta &= 1 - \frac{mk}{l^2} r_0 e^{-r_0/a}.
\end{aligned}
$$

This is the equation for a simple harmonic oscillator (with a constant shift) and frequency

$$\omega^2 = 1 - \frac{mk}{l^2} r_0^2 e^{-r_0/a} = 1 - \frac{r_0}{a}\frac{1}{1+\frac{r_0}{a}} = \frac{1}{1+\frac{r_0}{a}},$$

where we have used the definition of r_0 from (21). Now choose δ to be at maximum when $\theta = 0$, then the next maximum will occur when

$$\omega\theta = 2\pi \Rightarrow \theta = \frac{2\pi}{\omega} = 2\pi\left(1 + \frac{r_0}{2a}\right) + o\left(\left[\frac{r_0}{a}\right]^2\right).$$

Therefore the apsides advance by

$$\Delta\theta = \frac{\pi r_0}{a}$$

each revolution.