PHY 5246: Theoretical Dynamics, Fall 2015

Assignment # 6, Solutions

1 Graded Problems

Problem 1 (Goldstein 3.14)

1.a)

For a circular orbit, by conservation of energy we have

$$E = -\frac{k}{2r_0} = \frac{l^2}{2mr_0^2} - \frac{k}{r_0}$$

$$\frac{l^2}{2mr_0^2} = \frac{k}{2r_0} \Rightarrow r_0 = \frac{l^2}{mk} = r_{min}^{(c)}.$$

For a parabolic orbit with the same angular momentum l we have

$$E = 0 = \frac{l^2}{2mr_{min}^2} - \frac{k}{r_{min}} \Rightarrow r_{min}^{(p)} = \frac{l^2}{2mk}.$$

Thus we see

$$r_{min}^{(p)} = \frac{1}{2} r_{min}^{(c)}.$$

1.b)

For the circular orbit, with $v = r\dot{\theta}$ we have

$$\frac{1}{2}mr^2\dot{\theta}^2 - \frac{k}{r} = E = -\frac{k}{2r},$$

$$\frac{1}{2}mv^2 = \frac{k}{2r} \Rightarrow v_{(r)}^{(c)} = \sqrt{\frac{1}{m}\frac{k}{r}}.$$

For a parabolic orbit with the same r (using $v = \dot{r}^2 + r^2\dot{\theta}^2$) we can show

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{k}{r} = 0$$

$$\frac{1}{2}mv^2 = \frac{k}{r} \Rightarrow v_{(r)}^{(p)} = \sqrt{\frac{2}{m}\frac{k}{r}}.$$

This gives us the required result,

$$v_{(r)}^{(p)} = \sqrt{2} v_{(r)}^{(c)}.$$

Problem 2

The equation of the orbit for the force given in this problem is

$$\frac{\partial^2}{\partial\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=-\frac{mr^2}{l^2}F(r)=\frac{mk}{l^2}+\frac{m\lambda}{l^2 r} \ , \tag{1}$$

that can be written as

$$\frac{\partial^2}{\partial\theta^2}\left(\frac{1}{r}\right)+\left(1-\frac{m\lambda}{l^2}\right)\frac{1}{r}=\frac{mk}{l^2} \ . \tag{2}$$

This equation is of the form

$$\frac{\partial^2 u}{\partial\theta^2}+\omega^2 u=\frac{mk}{l^2} \ , \tag{3}$$

with $\omega^2=(1-m\lambda/l^2)$ and gives different solutions according to the different values of λ.

2.a) $\omega^2=(1-m\lambda/l^2)>0 \ \ \rightarrow \lambda<l^2/m$,
the equation is the same we encountered for the case of gravitational interaction, with the only difference that now $\omega^2\neq 1$. The solution of the equation is of the form:

$$u(\theta)=r^{-1}(\theta)=A\cos(\omega\theta)+\frac{mk}{l^2\omega^2} \ , \tag{4}$$

where we have chosen the arbitrary phase $\delta=0$. Depending on the value of the energy the orbit is either an ellipse, or a parabola or an hyperbola. In the case of bounded orbits (ellipse) we notice that the main difference with respect to the pure gravitational interaction is that the apsidal points are now reached at $\theta=0$ and $\theta=\pi/\omega>\pi$, which indicates a precession of the orbit.

2.b) $\omega^2=(1-m\lambda/l^2)=0 \ \ \rightarrow \lambda=l^2/m$,
the solution of the equation is simply

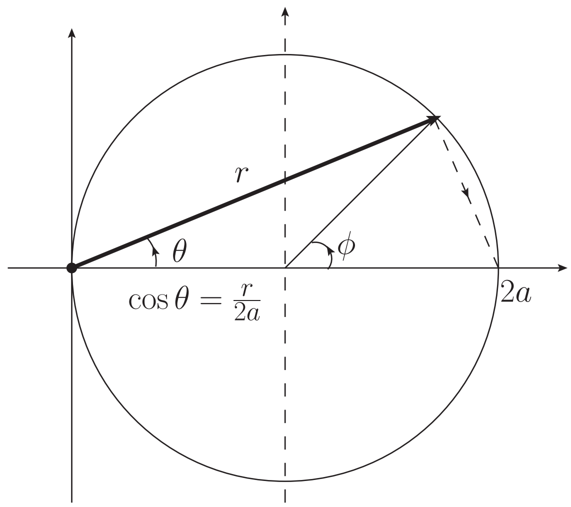
$$u(\theta)=r^{-1}(\theta)=\frac{mk}{2l^2}\theta^2+c_1\theta+c_2 \ , \tag{5}$$

showing that the radial distance decreases for increasing angles with an inverse quadratic law. The particle spirals in towards the center of force.

2.c) $\omega^2=(1-m\lambda/l^2)<0 \ \ \rightarrow \lambda>l^2/m$,
the solution is as for case **2.a)**, but expressed in terms of hyperbolic functions instead of harmonic functions, i.e.

$$u(\theta)=r^{-1}(\theta)=A\cosh(\omega\theta)+\frac{mk}{l^2\omega^2} \ , \tag{6}$$

showing that the radial distance decreases for increasing angles with a negative exponential law. In this case as well, the particle spirals in towards the center of force.



Problem 3 (Goldstein 3.13)

3.a)

We have for the equation of the orbit,

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{mr^2}{l^2}F(r). \tag{7}$$

Since $r = 2a\cos\theta$ (see figure), we can write

$$\begin{aligned}
\frac{d}{d\theta}\left(\frac{1}{r}\right) &= \frac{d}{d\theta}\left(\frac{1}{2a\cos\theta}\right) = \frac{1}{2a}\frac{d}{d\theta}\left(\frac{1}{\cos\theta}\right) \\
&= \frac{1}{2a}\frac{\sin\theta}{\cos^2\theta} \\
\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) &= \frac{1}{2a}\frac{d}{d\theta}\left(\frac{\sin\theta}{\cos^2\theta}\right) = \frac{1}{2a}\left(\frac{\cos^3\theta + \sin^2\theta 2\cos\theta}{\cos^4\theta}\right) \\
&= \frac{1}{2a}\left[\frac{1}{\cos\theta} + 2\frac{\sin^2\theta}{\cos^3\theta}\right] \\
&= \frac{1}{2a}\frac{1}{\cos\theta}\left[1 + \frac{2}{\cos^2\theta} - 2\right] \\
&= \frac{1}{2a\cos\theta}\left[\frac{2}{\cos^2\theta} - 1\right] = \frac{1}{r}\left[\frac{2\cdot 4a^2}{r^2} - 1\right] \\
&= \frac{1}{r}\left[\frac{8a^2}{r^2} - 1\right]
\end{aligned}$$

Now plugging this into (7), we find

$$\frac{1}{r}\left[\frac{8a^2}{r^2} - 1\right] + \frac{1}{r} = -\frac{mr^2}{l^2}F(r),$$

$$\longrightarrow F(r) = -\frac{l^2}{m}\frac{1}{r^2}\frac{8a^2}{r^3} = -\frac{8a^2l^2}{m}\frac{1}{r^5}.$$

3.b)

Finding the potential and kinetic energies:

$$\begin{aligned}
U(r) &= -\int_{\infty}^{r} dr' \frac{8a^2l^2}{m}\frac{1}{r'^5} = \frac{8a^2l^2}{m}\left(-\frac{1}{4}\frac{1}{r^4}\right) \\
&= -\frac{2a^2l^2}{m}\frac{1}{r^4} \\
T &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \\
&= \frac{1}{2}m\left[(-2a\sin\theta\dot{\theta})^2 + 4a^2\cos^2\theta\dot{\theta}^2\right] \\
&= \frac{1}{2}m\left[4a^2\dot{\theta}^2\right](\sin^2\theta + \cos^2\theta) = 2ma^2\dot{\theta}^2, \text{ now using } l = mr^2\dot{\theta} \\
&= 2ma^2\frac{l^2}{m^2r^4} = \frac{2a^2l^2}{m}\frac{1}{r^4} \\
E &= T + U = \frac{2a^2l^2}{m}\frac{1}{r^4} - \frac{2a^2l^2}{m}\frac{1}{r^4} = 0.
\end{aligned}$$

3.c)

From the figure we know that

$$\frac{dr}{d\theta} = -2a\sin\theta,$$

and so using the chain rule

$$\begin{aligned}
\frac{dr}{dt}\frac{dt}{d\theta} &= -2a\sin\theta, \text{ with } \frac{d\theta}{dt} = \dot{\theta} \text{ we have} \\
\frac{dr}{dt} &= -2a\sin\theta\dot{\theta} = -2a\sin\theta\frac{l}{mr^2} \\
&= -2a\sqrt{1-\cos^2\theta}\frac{l}{mr^2} = -2a\sqrt{1-\frac{r^2}{4a^2}}\frac{l}{mr^2} \\
&= -\sqrt{4a^2 - r^2}\frac{l}{mr^2}.
\end{aligned}$$

Now we integrate this equation over one orbit to find the period. To avoid any possible problems with multiple-valued functions, we will integrate from $r_{max} = 2a$ to $r_{min} = 0$ and multiply the result by two.

$$\begin{aligned}
dt &= -\frac{m}{l}\frac{r^2dr}{\sqrt{4a^2 - r^2}} \\
T &= 2\int_{2a}^{0}\left(-\frac{m}{l}\frac{r^2dr}{\sqrt{4a^2 - r^2}}\right) \\
&= -\frac{2m}{l}\int_{2a}^{0}\frac{r^2dr}{\sqrt{4a^2 - r^2}} \\
&= -\frac{2m}{l}\left[-\frac{r}{2}\sqrt{4a^2 - r^2} + \frac{4a^2}{2}\sin^{-1}\left(\frac{r}{2a}\right)\right]_{2a}^{0} \\
&= \frac{2m}{l}\left[\frac{4a^2}{2}\frac{\pi}{2}\right] = \frac{2m}{l}a^2\pi = 2\pi\frac{ma^2}{l}.
\end{aligned}$$

3.d)

From the definition of ϕ in the figure we see that $\phi = 2\theta$, so we can write our Cartesian coordinates as

$$\begin{cases} x &= a + a\cos\phi \\ y &= a\sin\phi \end{cases}$$

$$\downarrow$$

$$\begin{cases} x &= a + a\cos 2\theta &= a(2\cos^2\theta - 1 + 1) &= 2a\cos^2\theta \\ y &= a\sin 2\theta &= 2a\sin\theta\cos\theta \end{cases}$$

Take a derivative with respect to time of these to find $\dot{x}$ and $\dot{y}$:

$$\begin{aligned} \dot{x} &= -2a2\cos\theta\sin\theta\dot{\theta} = -4a\frac{l}{mr^2}\cos\theta\sin\theta \\ &= -\frac{2l}{mr^2}r\sqrt{1-\frac{r^2}{4a^2}} = -\frac{2l}{m}\frac{1}{r}\sqrt{1-\frac{r^2}{4a^2}} \\ \dot{y} &= 2a\dot{\theta}\cos 2\theta = 2a\frac{l}{mr^2}(2\cos^2\theta - 1) \\ &= 2a\frac{l}{mr^2}\frac{1}{a}r^2 - \frac{2al}{mr^2} = \frac{l}{ma}\left[1 - \frac{2a^2}{r^2}\right]. \end{aligned}$$

Here we have used $r = 2a\cos\theta$ and $4a^2 - r^2 = 2a\sin\theta$. Now we can see that as $r \to 0$,

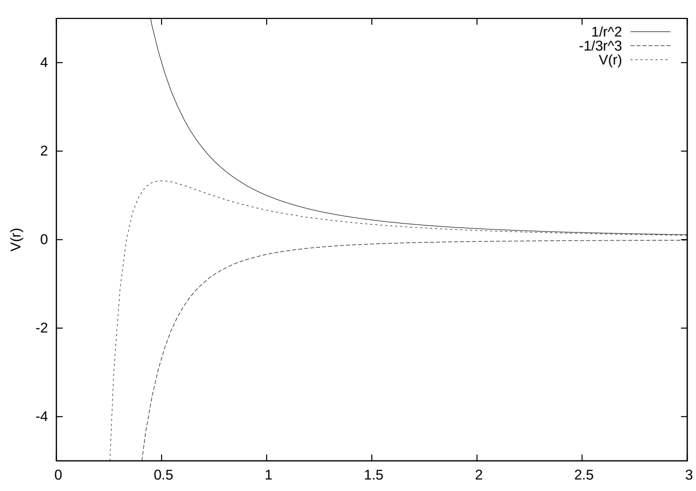
$$\begin{cases} \dot{x} & \to & -\infty \\ \dot{y} & \to & -\infty \end{cases} \Rightarrow v = \sqrt{\dot{x}^2 + \dot{y}^2} \to \infty.$$

Problem 4

The effective potential is

$$V'(r) = \frac{l^2}{2mr^2} - \frac{C}{3r^3} \ , \tag{8}$$

it becomes infinitely negative for $r \to 0$, it has a positive maximum for $r = r_0$ (see **(1.a)**), and vanishes for $r \to \infty$, as illustrated in the plot.



4.a)

The extrema of the effective potential are found by searching the zeros of the derivative of $V'(r)$ with respect to r,

$$\frac{dV'(r)}{dr} = -\frac{l^2}{mr^3} + \frac{C}{r^4} = \frac{1}{r^3}\left[-\frac{l^2}{m} + \frac{C}{r}\right] \ , \tag{9}$$

i.e.

$$\frac{dV'(r)}{dr} = 0 \Rightarrow r = r_0 = \frac{Cm}{l^2} \ . \tag{10}$$

Both from the plot of $V'(r)$ and from the fact that

$$\left.\frac{d^2V'(r)}{dr^2}\right|_{r=r_0} = \frac{3C}{r_0^5} - \frac{4C}{r_0^5} = -\frac{C}{r_0^5} < 0 \ , \tag{11}$$

we see that $r = r_0$ is a maximum of the effective potential. The maximum value of the effective potential is

$$V'(r)_{\max} = V'(r_0) = \frac{l^2}{2mr_0^2} - \frac{C}{3r_0^3} = \frac{C}{2}\frac{1}{r_0^3} - \frac{C}{3}\frac{1}{r_0^3} = \frac{C}{6}\frac{1}{r_0^3} \ . \tag{12}$$

4.b)

If a particle comes in with energy $E > V'(r)_{\max}$ it can continue all the way to $r = 0$, i.e. it is captured by the center of force. The condition on the energy gives automatically a condition on the impact parameter since

$$E_{\min} = V'(r)_{\max} = \frac{l^6}{6C^2m^3} = \frac{(mv_0b)^6}{6C^2m^3} = \frac{m^3v_0^6b^6}{6C^2} \ , \tag{13}$$

and then

$$E > E_{\min} \quad \Rightarrow \quad \frac{1}{2}mv_0^2 > \frac{m^3v_0^6b^6}{6C^2} \quad \Rightarrow \quad b < b_{\max} = \left(\frac{3C^2}{m^2v_0^4}\right)^{1/6} \ . \tag{14}$$

Consequently the cross section for capture is

$$\sigma_{\text{capture}} = \pi b_{\max}^2 = \pi\left(\frac{3C^2}{m^2v_0^4}\right)^{1/3} \ . \tag{15}$$

2 Non-graded Problems

Problem 5 (Goldstein 3.10)

The planet with mass M is moving on a highly eccentric elliptic orbit, so its energy is

$$E_0 = -\frac{k}{2a},$$

where note that $\epsilon \sim 1 \Rightarrow E \sim 0$, but not quite. In other words you need just a small change ΔE to get to $E = 0$. At the aphelion (where it has velocity v_0, all in the tangent direction) it is hit by a comet (with mass $m << M$) moving with velocity v_c in the same tangent direction. Thus the collision is head-on with both objects moving in the same direction. The collision is assumed to be completely inelastic, *i.e.* the two objects stick together forming a new object of mass $(M+m)$. Linear momentum is conserved while kinetic energy is not.

Conservation of linear momentum gives

$$\begin{aligned} Mv_0 + mv_c &= (M+m)(v_0+\delta v) \\ v_c &= \frac{1}{m}\left[(M+m)(v_0+\delta v) - Mv_0\right] \\ &= \frac{1}{m}\left[M\delta v + mv_0 + o(m\delta v)\right] \\ &\approx \frac{M}{m}\delta v + v_0. \end{aligned} \tag{16}$$

In the second to last line we have kept $o(m/M)$ but are going to discard $o(m/M\delta v)$ since δv is a small quantity because v_0 is almost small enough to make $E=0$ (but not quite!). The energy is not conserved, and in fact the collision must increase the energy from the initial value E_0 ($E_0<0$) to $E=0$ in order for the resulting orbit to be parabolic. So, the comet has to have a kinetic energy

$$T_c^{(min)} = \frac{1}{2}mv_c^2 \tag{17}$$

so that the final energy is 0:

$$\begin{aligned} E &= \frac{1}{2}v_f^2 - \frac{k}{r} = \frac{1}{2}(M+m)(v_0+\delta v)^2 - \frac{k}{r} \\ &= \frac{1}{2}Mv_0^2 + Mv_0\delta v + \frac{1}{2}mv_0^2 + o(\delta v^2) + o(m\delta v) \\ &= E_0 + Mv_0\delta v + \frac{1}{2}mv_0^2, \end{aligned}$$

where $E_0 = \frac{1}{2}Mv_0^2 - \frac{k}{r}$. Setting $E=0$ we get

$$\delta v = -\frac{E_0}{Mv_0} - \frac{1}{2}\frac{m}{M}v_0.$$

Plugging this into (16) we find

$$\begin{aligned} v_c &= \frac{M}{m}\left(-\frac{E_0}{Mv_0} - \frac{1}{2}\frac{m}{M}v_0\right) + v_0 \\ &= -\frac{E_0}{mv_0} - \frac{1}{2}v_0 + v_0 = -\frac{E_0}{mv_0} + \frac{1}{2}v_0. \end{aligned}$$

Plugging this into (17) we finally find

$$\begin{aligned} T_c^{(min)} &= \frac{1}{2}m\left(-\frac{E_0}{mr_0} + \frac{1}{2}v_0\right)^2 \\ &= \frac{1}{2}m\frac{E_0^2}{m^2v_0^2} + \frac{1}{2}m\left(-\frac{E_0}{m}\right) + \frac{1}{2}m\frac{1}{4}v_0^2 \\ &= \frac{1}{2}\frac{E_0^2}{mv_0^2} - \frac{E_0}{2} + \frac{1}{8}mv_0^2. \end{aligned}$$

Now we can finally derive v_0 in terms of the constant parameters of the orbit, the mass of planet and k. From angular momentum conservation we have that (at the aphelion, and using $\epsilon = 1-\alpha$)

$$\begin{aligned} l &= Mr_{max}v_0 = Ma(1+\epsilon)v_0 = Ma(2-\alpha)v_0 \\ v_0^2 &= \frac{l^2}{M^2a^2(2-\alpha)^2} = \frac{aMk(1-\epsilon^2)}{M^2a^2(2-\alpha)^2} \\ &= \frac{aMk(1-(1-\alpha)^2)}{M^2a^2(2-\alpha)^2} \approx \frac{\alpha k}{2Ma}. \end{aligned}$$

So now we have

$$\begin{aligned} T_c^{(min)} &= \frac{1}{2}\frac{E_0^2}{m\frac{\alpha k}{2Ma}} - \frac{E_0}{2} + \frac{1}{8}m\frac{\alpha k}{2Ma} \\ &= \frac{M}{m}\frac{k}{4a\alpha} + \frac{k}{4a} + \frac{1}{16}\frac{m}{M}\frac{\alpha k}{a} \\ &= \frac{k}{4a}\left(\frac{1}{\alpha}\frac{M}{m} + 1\right) + \frac{m}{M}\alpha\frac{k}{16a}. \end{aligned}$$

Note that the second term is small compared to the first since $\alpha << 1$ and $m/M << 1$. Therefore

$$T_c^{(min)} \approx \frac{1}{\alpha}\frac{M}{m}\frac{k}{4a}.$$