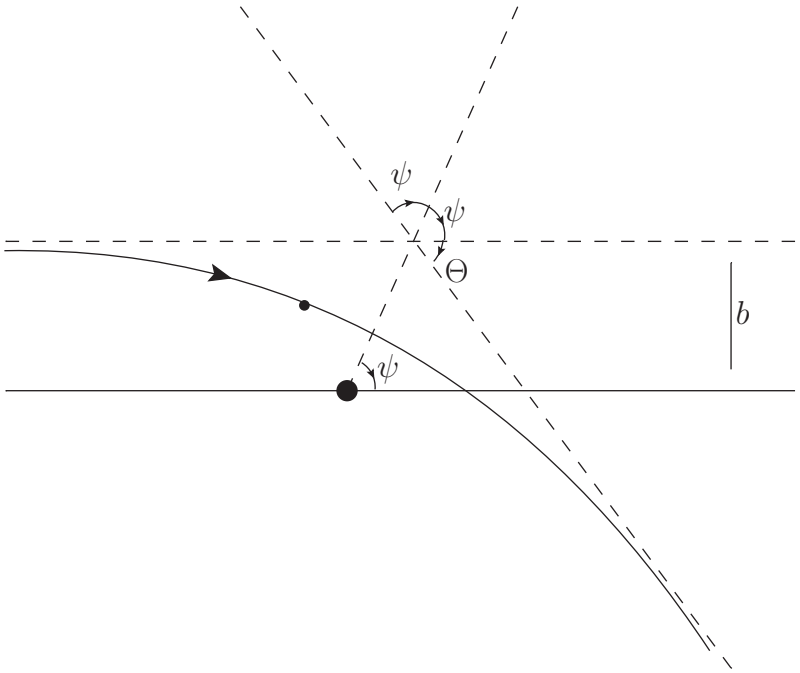
PHY 5246: Theoretical Dynamics, Fall 2015

Assignment # 7, Solutions

1 Graded Problems

Problem 1



(1.a)

The scattering angle satisfies the relation

$$\Theta = \pi - 2\psi \ , \tag{1}$$

where ψ is the angle between the direction of the incoming asymptote and the periapsis (the direction of closest approach), and can be obtained from the equation of the orbit

$$\theta = \int_{r_0}^{r} \frac{dr'}{(r')^2 \sqrt{\frac{2Em}{l^2} - \frac{2mV}{l^2} - \frac{1}{(r')^2}}} + \theta_0 \ , \tag{2}$$

setting $\theta_0 = \pi$ for $r_0 = \infty$ (the incoming direction), such that $\theta = \pi - \theta$ for $r = r_{\min}$, i.e.

$$\psi = \int_{r_{\min}}^{\infty} \frac{dr}{r^2 \sqrt{\frac{2Em}{l^2} - \frac{2mV}{l^2} - \frac{1}{r^2}}} \ . \tag{3}$$

The equation of the hyperbolic orbit is then

$$\frac{1}{r} = \frac{mk}{l^2} \left(1 + \epsilon \cos(\theta - \pi)\right) = \frac{mk}{l^2} \left(1 - \epsilon \cos\theta\right) \ , \tag{4}$$

where $\epsilon = \sqrt{1 + 2El^2/(mk^2)} > 1 \leftrightarrow E > 0$ (eccentricity of the hyperbolic orbit) and ψ is defined by the limit

$$r \to \infty \quad \to \quad \cos\psi = \frac{1}{\epsilon} \ . \tag{5}$$

In terms of the scattering angle the previous condition becomes

$$\cos\left(\frac{\pi}{2} - \frac{\Theta}{2}\right) = \sin\frac{\Theta}{2} = \frac{1}{\epsilon} \ , \tag{6}$$

which implies

$$\tan^2\frac{\Theta}{2} = \frac{1}{\epsilon^2 - 1} \quad \to \quad \cot^2\frac{\Theta}{2} = \epsilon^2 - 1 = \frac{2El^2}{mk^2} = \frac{m^2 v_0^4 b^2}{k^2} \tag{7}$$

and finally

$$\cot\frac{\Theta}{2} = \frac{mv_0^2 b}{k} = \frac{v_0^2 b}{GM} \quad \Rightarrow \quad b = \frac{1}{\gamma}\cot\frac{\Theta}{2} \ , \tag{8}$$

for $k = GMm$ and $\gamma = v_0^2/(GM)$.

(1.b)

Using the notation of your book,

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\Theta}\left|\frac{db}{d\Theta}\right| \ . \tag{9}$$

Using the result in **(2.a)**, we can write it as

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\Theta}\frac{1}{2\gamma}\frac{1}{\sin^2\frac{\Theta}{2}} = \frac{1}{\gamma}\frac{\cot\frac{\Theta}{2}}{2\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}}\frac{1}{2\gamma}\frac{1}{\sin^2\frac{\Theta}{2}} = \frac{1}{4\gamma^2\sin^4\frac{\Theta}{2}} \ . \tag{10}$$

(1.c)

For $\Theta = 180°$, i.e. for the case of *backward scattering*, the differential scattering cross section is

$$\frac{d\sigma}{d\Omega} \simeq \frac{1}{4\gamma^2} \simeq \frac{1}{v_0^4} \ , \tag{11}$$

and

$$\begin{aligned} \frac{d\sigma}{d\Omega} \quad &\to \quad \infty \quad \text{for} \quad v_0 \to 0 \ , \\ \frac{d\sigma}{d\Omega} \quad &\to \quad 0 \quad \text{for} \quad v_0 \to \infty \ . \end{aligned} \tag{12}$$

Indeed, for $v_0 \to 0$ (i.e. if particles approach with very low speed) we have that $E \to 0$, i.e. the orbit degenerates into a parabolic orbit and all particles, after having approached the center of force, move back from where they approached from. On the other hand, if $v_0 \to \infty$ (i.e. if particles approach with very high speed), their kinetic energy is very large, the total energy of the orbit is very mildly affected by the center of force potential energy, and the particles' orbits are very mildly deflected, such that almost no particles are deflected backward.

(1.d)

For $\Theta \simeq 0$, i.e. for the case of *forward scattering*, we see that

$$\frac{d\sigma}{d\Omega} \to \infty \ , \tag{13}$$

i.e. the cross section for forward scattering seems to be infinite. This is due to the fact that all impact parameters can contribute to the cross section, up to infinity. Of course, the larger the impact parameter of a given trajectory, the milder the deviation of of trajectory from the initial direction. All particles coming in with very large impact parameter are scattered in the forward direction, and, if all impact parameters contribute, the cross section for forward scattering is infinite.

The only way to prevent such an unphysical situation is to cut off the impact parameter. Is this a trick? Not quite. Indeed in nature all scattering problems have this property. Objects scattering with large impact parameter do not feel the center of force because this is screened by other interactions (think for instance to the case of Rutherford scattering and the effect of electrons in screening the nuclei of atoms if the incoming particles are at a distance larger than the atomic distance), and that prevents any forward scattering physical cross section from being infinite.

(1.e)

For Coulomb interaction the only difference is that k in all previous formulas can be replaced by Kq_1q_2 (for K a given constant, in this case proportional to the electron charge square) and we get $\gamma_e = mv_0^2/(Kq_1q_2) = 2E^2/(Kq_1q_2)$, such that

$$\frac{d\sigma}{d\Omega} = \frac{K^2 q_1^2 q_2^2}{16E^2 \sin^4 \frac{\Theta}{2}} \ . \tag{14}$$

Problem 2 (Goldstein 3.31)

Calculate the potential due to a force $f(r) = kr^{-3}$.

$$V(r) = -\int_\infty^r dr' \frac{k}{r^3} = -\left(-\frac{1}{2}\frac{k}{r^2}\right)\Bigg|_\infty^r = \frac{k}{2r^2}.$$

Using equation (3.96) in Goldstein we can calculate the deflection angle with impact parameter s as

$$\begin{aligned} \Theta(s) &= \pi - 2\int_{r_m}^\infty \frac{sdr}{r\sqrt{r^2\left(1-\frac{V(r)}{E}\right) - s^2}} \\ &= \pi - 2\int_{r_m}^\infty \frac{sdr}{r\sqrt{r^2 - \left(\frac{k}{2E} + s^2\right)}}. \end{aligned}$$

Notice that from conservation of energy:

$$E = E(\infty) = \frac{1}{2}mv_0^2 = E(r_m) = \frac{k}{2r_m^2} + \frac{1}{2}mr_m^2\dot{\theta}^2$$

$$
\begin{aligned}
&= \frac{k}{2r_m^2} + \frac{1}{2} m r_m^2 \frac{l^2}{m^2 r_m^4} \\
&= \frac{k}{2r_m^2} + \frac{1}{2}\frac{s^2 2mE}{m r_m^4 2} = \frac{1}{r_m^2} E\left(\frac{k}{2E} + s^2\right) \longrightarrow 1 = \frac{1}{r_m^2}\left(\frac{k}{2E} + s^2\right) \\
&\Rightarrow r_m^2 = \frac{k}{2E} + s^2.
\end{aligned}
$$

Therefore this integral is

$$
\begin{aligned}
\Theta(s) &= \pi - 2s \int_{r_m}^{\infty} \frac{dr/r}{\sqrt{r^2 - r_m^2}} \\
&= \pi - 2s \int_{r_m}^{\infty} \frac{dr/r^2}{\sqrt{1 - \left(\frac{r_m}{r}\right)^2}} \\
&= \pi - \frac{2s}{r_m}\left[\arccos\left(\frac{r_m}{r}\right)\right]_{r_m}^{\infty} = \pi - \frac{s}{r_m}\pi \\
&= \pi\left(1 - \frac{s}{r_m}\right) = \pi\left[1 - \frac{s}{\sqrt{\frac{k}{2E} + s^2}}\right]. \qquad (15)
\end{aligned}
$$

Now in terms of $x = \Theta/\pi$ the above becomes

$$
x = 1 - \frac{s}{\sqrt{\frac{k}{2E} + s^2}} \longrightarrow (1-x)\sqrt{\frac{k}{2E} + s^2} = s
$$

$$
(1-x)^2\left(\frac{k}{2E} + s^2\right) = s^2 \longrightarrow s^2\left[1 - (1-x)^2\right] = \frac{k}{2E}(1-x)^2.
$$

Therefore we can solve for the impact parameter

$$
s = \sqrt{\frac{k}{2E}}\frac{1-x}{\sqrt{1-(1-x)^2}}.
$$

The differential cross section can be expressed in terms of the scattering angle by eq (3.93) in Goldstein,

$$
\sigma(\Theta)d\Theta = \frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right| d\Theta. \qquad (16)
$$

From (15) we can calculate:

$$
\begin{aligned}
\frac{d\Theta}{ds} &= -\pi \frac{\sqrt{\frac{k}{2E} + s^2} - \frac{1}{2} s \frac{2s}{\sqrt{\frac{k}{2E} + s^2}}}{\frac{k}{2E} + s^2} \\
&= -\pi \frac{\frac{k}{2E} + s^2 - s^2}{\left(\frac{k}{2E} + s^2\right)^{3/2}} = -\frac{\pi k}{2E}\frac{1}{\left(\frac{k}{2E} + s^2\right)^{3/2}},
\end{aligned}
$$

and inverting this we find

$$
\frac{ds}{d\Theta} = -\frac{2E}{\pi k}\left(\frac{k}{2E} + s^2\right)^{3/2}.
$$

Plugging this into (16) we get the desired result:

$$\begin{aligned}\sigma(\Theta)d\Theta &= \frac{s}{\sin(\pi x)}\frac{2E}{\pi k}\left(\frac{k}{2E}+s^2\right)^{3/2}d\Theta \\ &= \sqrt{\frac{k}{2E}}\frac{1-x}{\sqrt{1-(1-x)^2}}\frac{2E}{k}\left(\frac{k}{2E}\right)^{3/2}\left[1+\frac{(1-x)^2}{1-(1-x)^2}\right]^{3/2}\frac{dx}{\sin(\pi x)} \\ &= \frac{k}{2E}\frac{1-x}{(1-(1-x)^2)^2}\frac{dx}{\sin(\pi x)} = \frac{k}{2E}\frac{1-x}{x^2(2-x)^2}\frac{dx}{\sin(\pi x)}.\end{aligned}$$

Problem 3

Starting from the relation between the velocity of the incident particle after scattering in the laboratory frame ($\mathbf{v_1}$) and in the center-of-mass frame ($\mathbf{v_1'}$),

$$\mathbf{v_1} = \mathbf{V} + \mathbf{v_1'} \ , \tag{17}$$

where $\mathbf{V}$ is the velocity of the center of mass, we can derive that

$$v_1\cos\theta = v_1'\cos\Theta + V \ \ \rightarrow \ \ \cos\theta = \frac{v_1'\cos\Theta + V}{v_1} \ , \tag{18}$$

as one obtains by projecting Eq. (17) along the direction of the approaching incident particle, and

$$v_1^2 = (v_1')^2 + V^2 + 2v_1'V\cos\Theta \ \ \rightarrow \ \ \cos\Theta = \frac{v_1^2 - (v_1')^2 - V^2}{2v_1'V} \ , \tag{19}$$

as obtained by squaring Eq. (17). Substituting Eq. (19) into Eq. (18) we can write that

$$\cos\theta = \frac{v_1^2 - (v_1')^2 + V^2}{2v_1V} \ . \tag{20}$$

Furthermore, conservation of momentum (and assuming that the second particle is initially at rest) tells us that

$$(m_1+m_2)\mathbf{V} = m_1\mathbf{v_0} \ \ \rightarrow \ \ V = \frac{m_1}{m_1+m_2}v_0 \ , \tag{21}$$

while, using the kinematic of a two-body system in the center-of-mass frame, we can write that

$$\mathbf{r_1'} = -\frac{m_2}{m_1+m_2}\mathbf{r} \ \ \rightarrow \ \ v_1' = \dot{r}_1' = \frac{m_2}{m_1+m_2}v \ , \tag{22}$$

where $v = \dot{r}$ is the relative velocity after the collision. Finally, we can trade velocities for energies by using that:

- the energy of the incident particle before scattering in the laboratory frame (E_0) is

$$E_0 = \frac{1}{2}m_1v_0^2 \ \ \rightarrow \ \ v_0 = \sqrt{\frac{2E_0}{m_1}} \ , \tag{23}$$

- the energy of the incoming particle after scattering in the laboratory frame (E_1) is

$$E_1 = \frac{1}{2} m_1 v_1^2 \;\; \rightarrow \;\; v_1 = \sqrt{\frac{2E_1}{m_1}} \; , \tag{24}$$

- from conservation of energy we have that

$$E_i - Q = E_0 - Q = E_f = E_{\rm CM} + \frac{1}{2}\mu v^2 \; , \tag{25}$$

where $E_{\rm CM} = \frac{1}{2}(m_1 + m_2)V^2$, $\mu = \frac{m_1 m_2}{(m_1+m_2)}$ is the reduced mass of the system, and $v = \dot{r}$ is the relative velocity after the collision. Using simple manipulations Eq. (25) gives

$$\frac{1}{2}\mu v_0^2 - Q = \frac{1}{2}\mu v^2 \;\; \rightarrow \;\; v^2 = v_0^2 - \frac{2}{\mu} Q \; . \tag{26}$$

Substituting Eqs. (21)-(26) into Eq. (20) we get,

$$\cos\theta = \frac{m_1 + m_2}{2m_1}\sqrt{\frac{E_1}{E_0}} + \frac{m_1 - m_2}{2m_1}\sqrt{\frac{E_0}{E_1}} + \frac{m_2 Q}{2m_1\sqrt{E_0 E_1}} \; . \tag{27}$$

2 Non-graded Problems

Problem 5 (Goldstein 3.32)

The potential has the form

$$\left\{ \begin{array}{rcll} V & = & 0 & r > a \\ & = & -V_0 & r \leq a \end{array} \right.$$

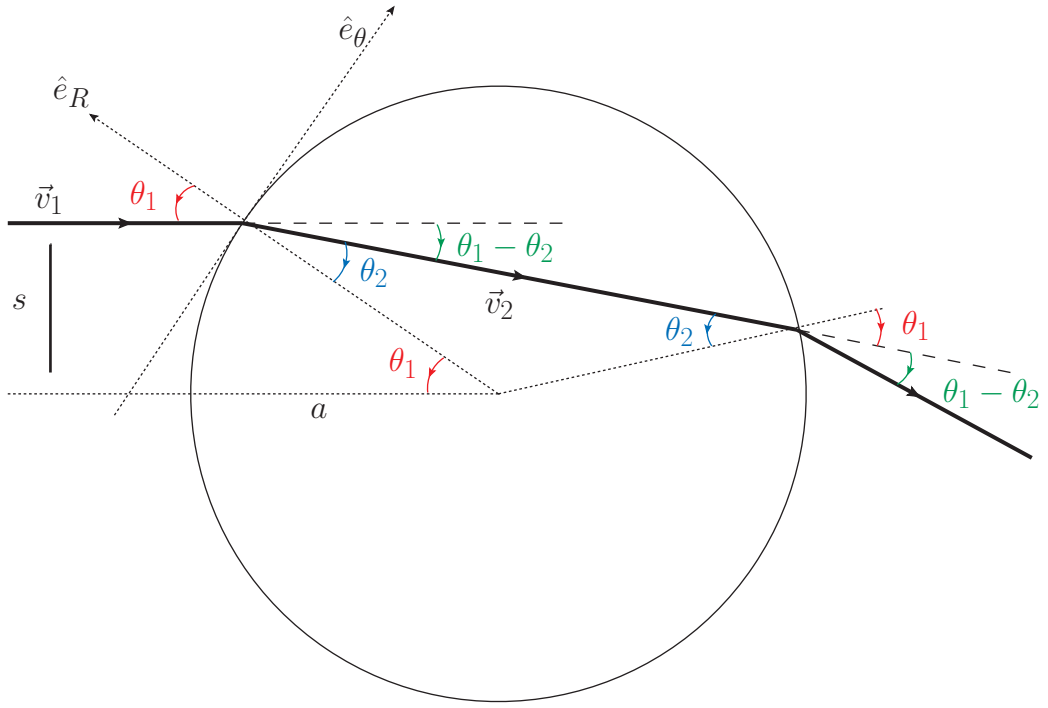
First consider what happens on the interface with the spherical surface that separates the region with $V = 0$ from the region with $V = -V_0$. Given the geometry of the problem, it seems obvious to use spherical coordinates. However, since the problem has azimuthal symmetry, we can use polar coordinates.

Notice that the force acting on the incoming particle is all in the $\hat{\mathbf{e}}_r$ direction (given $V(r)$). Thus the linear momentum in the $\hat{\mathbf{e}}_\theta$ direction is conserved. We can then write

$$\begin{array}{rcl} \vec{\mathbf{v}}_1 & = & -v_1 \cos\theta_1 \hat{\mathbf{e}}_r + v_1 \sin\theta_1 \hat{\mathbf{e}}_\theta \\ \vec{\mathbf{v_2}} & = & -v_2 \cos\theta_2 \hat{\mathbf{e}}_r + v_2 \sin\theta_2 \hat{\mathbf{e}}_\theta \\ & & \Rightarrow v_1 \sin\theta_1 = v_2 \sin\theta_2 \text{ or } v_2 = \dfrac{\sin\theta_1}{\sin\theta_2} v_1. \end{array}$$

We can also use conservation of energy for the incoming particle,

$$\begin{array}{rcl} E & = & \dfrac{1}{2} m v_1^2 = \dfrac{1}{2} m v_2^2 - V_0 \\ & = & \dfrac{1}{2} m v_1^2 = \dfrac{1}{2} m \dfrac{\sin^2\theta_1}{\sin^2\theta_2} v_1^2 - V_0 \\ & & \Rightarrow \dfrac{\sin^2\theta_1}{\sin^2\theta_2} = 1 + \dfrac{1}{\frac{1}{2} m v_1^2} V_0 = 1 + \dfrac{V_0}{E}. \end{array}$$



Thus this has exactly the form of Snell's law! The refractive index is

$$n = \frac{\sin\theta_1}{\sin\theta^2} = \sqrt{\frac{E+V_0}{E}}. \tag{28}$$

Now refer to figure 2. The scattering angle is

$$\Theta = 2(\theta_1 - \theta_2)$$

and the impact parameter s satisfies

$$s = a\sin\theta_1.$$

We need to derive $s = s(\Theta)$ and insert it into the standard formula for the differential cross-section

$$\sigma(\Theta) = \frac{s}{\sin\theta}\left|\frac{ds}{d\Theta}\right|. \tag{29}$$

Using (28) we have

$$n = \frac{\sin\theta_1}{\sin\theta_2} = \frac{\sin\theta_1}{\sin(\theta_1 - \frac{\Theta}{2})}.$$

Inverting this and using some trig. identities we find

$$\frac{1}{n} = \frac{\sin\theta_1\cos\frac{\Theta}{2} - \cos\theta_1\sin\frac{\Theta}{2}}{\sin\theta_1} = \cos\frac{\Theta}{2} - \cot\theta_1\sin\frac{\Theta}{2}. \tag{30}$$

Now since $\sin\theta_1 = s/a$ we can write

$$\cot\theta_1 = \frac{\cos\theta_1}{\sin\theta_1} = \frac{\sqrt{1-\left(\frac{s}{a}\right)^2}}{s/a} = \sqrt{\left(\frac{a}{s}\right)^2 - 1}.$$

Now inserting this relationship into (30), we can solve for s:

$$\frac{1}{n} = \cos\frac{\Theta}{2} - \sqrt{\left(\frac{a}{s}\right)^2 - 1}\sin\frac{\Theta}{2},$$

$$\sqrt{\left(\frac{a}{s}\right)^2 - 1} = \frac{\cos\frac{\Theta}{2} - \frac{1}{n}}{\sin\frac{\Theta}{2}},$$

$$\left(\frac{a}{s}\right)^2 = 1 + \frac{\left(\cos\frac{\Theta}{2} - \frac{1}{n}\right)^2}{\sin^2\frac{\Theta}{2}},$$

$$\left(\frac{s}{a}\right)^2 = \frac{\sin^2\frac{\Theta}{2}}{\sin^2\frac{\Theta}{2} + \left(\cos\frac{\Theta}{2} - \frac{1}{n}\right)^2}.$$

$$\Rightarrow s^2 = \frac{a^2\sin^2\frac{\Theta}{2}}{1 + \frac{1}{n^2} - \frac{2}{n}\cos\frac{\Theta}{2}} = \frac{a^2n^2\sin^2\frac{\Theta}{2}}{n^2 + 1 - 2n\cos\frac{\Theta}{2}}.$$

now we calculate the quantity

$$\begin{aligned} 2s\frac{ds}{d\Theta} &= \frac{a^2n^2}{\left(n^2 + 1 - 2n\cos\frac{\Theta}{2}\right)^2}\left\{\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}(n^2+1) - n\sin\frac{\Theta}{2}\left(2\cos^2\frac{\Theta}{2} + \sin^2\frac{\Theta}{2}\right)\right\} \\ &= \frac{a^2n^2\sin\frac{\Theta}{2}}{\left(n^2 + 1 - 2n\cos\frac{\Theta}{2}\right)^2}\left(n\cos\frac{\Theta}{2} - 1\right)\left(n - \cos\frac{\Theta}{2}\right). \end{aligned}$$

Now plugging this last expression into (29) we find the required result,

$$\begin{aligned} \sigma(\Theta) &= \frac{s}{2\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}}\frac{1}{2s}\frac{a^2n^2\sin\frac{\Theta}{2}}{\left(n^2 + 1 - 2n\cos\frac{\Theta}{2}\right)^2}\left(n\cos\frac{\Theta}{2} - 1\right)\left(n - \cos\frac{\Theta}{2}\right) \\ &= \frac{a^2n^2}{4\cos\frac{\Theta}{2}}\frac{\left(n\cos\frac{\Theta}{2} - 1\right)\left(n - \cos\frac{\Theta}{2}\right)}{\left(n^2 + 1 - 2n\cos\frac{\Theta}{2}\right)^2}. \end{aligned}$$