PHY 5246: Theoretical Dynamics, Fall 2015
Assignment # 8, Solutions

1 Graded problems

1. In coordinates (θ_1, θ_2) (see figure), we have

$$\begin{aligned} T &= \frac{1}{2}m(b\dot{\theta}_1)^2 + \frac{1}{2}m(b\dot{\theta}_2)^2 \\ V &= mgb(1-\cos\theta_1) + mgb(1-\cos\theta_2) + \frac{1}{2}k(b\sin\theta_1 - b\sin\theta_2)^2. \end{aligned}$$

Note the equilibrium length of the spring does not appear because of the manner in which the problem is presented (unstreched in the equilibrium position). Now we expand these coordinates about the equilibrium $\theta_{0,1} = \theta_{0,2} = 0$, and using the small displacements (η_1, η_2) we can make the following approximations:

$$\sin\theta_i = \frac{\eta_i}{b} \approx \theta, \quad \cos\theta_i \approx 1 - \frac{1}{2}\theta_i^2 = 1 - \frac{1}{2}\frac{\eta_i^2}{b^2}.$$

Now our energies are

$$\begin{aligned} T &= \frac{1}{2}m\dot{\eta}_1^2 + \frac{1}{2}m\dot{\eta}_2^2 \\ V &= mgb\left[1 - \left(1 - \frac{1}{2}\frac{\eta_1^2}{b^2}\right)\right] + mgb\left[1 - \left(1 - \frac{1}{2}\frac{\eta_2^2}{b^2}\right)\right] + \frac{1}{2}k(\eta_1 - \eta_2)^2, \end{aligned}$$

and our Lagrangian is

$$L = \frac{1}{2}m(\dot{\eta}_1^2 + \dot{\eta}_2^2) - \frac{mg}{2b}(\eta_1^2 + \eta_2^2) - \frac{1}{2}k(\eta_1 - \eta_2)^2.$$

Our equations of motion (via Euler-Lagrange) are

$$m\ddot{\eta}_1 + \left(k + \frac{mg}{b}\right)\eta_1 - k\eta_2 = 0$$

$$m\ddot{\eta}_2 + \left(k + \frac{mg}{b}\right)\eta_2 - k\eta_1 = 0.$$

Using an ansatz $\eta_i = a_i e^{i\omega t}$ these equations are

$$-m\omega^2 a_1 + \left(k + \frac{mg}{b}\right)a_1 - ka_2 = 0 \tag{1}$$

$$-m\omega_2 a_2 + \left(k + \frac{mg}{b}\right)a_2 - ka_1 = 0. \tag{2}$$

These equations will have a nontrivial solution only if the determinant vanishes, which gives us a condition on the eigenfrequencies ω.

$$\begin{vmatrix} k + \frac{mg}{b} - m\omega^2 & -k \\ -k & k + \frac{mg}{b} - m\omega^2 \end{vmatrix} \longrightarrow \left(k + \frac{mg}{b} - m\omega^2\right)^2 - k^2 = 0.$$

$$k + \frac{mg}{b} - m\omega^2 = \pm k \longrightarrow \omega^2 = \frac{k + \frac{mg}{b} \pm k}{m} \longrightarrow \omega = \pm\sqrt{\frac{k + \frac{mg}{b} \pm k}{m}}.$$

Thus, as expected we have two different eigenfrequencies

$$\omega_1 = \pm\sqrt{\frac{g}{b}}, \qquad \omega_2 = \pm\sqrt{\frac{g}{b} + 2\frac{k}{m}},$$

where the $\pm$ would be our positive and negative solutions if we wanted to do a full solution with initial conditions. For our purposes we only want the normal modes, so we define normal coordinates as

$$\eta_i(t) = \sum_j a_{ij}\zeta_j(t), \qquad \zeta_j(t) = e^{i\omega_j t}.$$

To find the normal coordinates we first plug ω_1 into equation (1)(setting $a_i = a_{i1}$ to match our normal coordinates), and find

$$\left(k + \frac{mg}{b} - m\frac{g}{b}\right) a_{11} - k a_{21} = 0 \longrightarrow a_{11} = a_{21}.$$

Now plugging ω_2 into (2) (and with $a_i = a_{i2}$) we find

$$-m\left(\frac{2k}{m} + \frac{g}{b}\right) a_{12} + \left(k + \frac{mg}{b}\right) a_{12} - k a_{22} = 0 \longrightarrow a_{12} = -a_{22}.$$

We could normalize these eigenvectors by simply making their length equal to 1:

$$\begin{aligned} a_{11}^2 + a_{21}^2 &= 1 \rightarrow a_{11} = a_{21} = \frac{1}{\sqrt{2}} \\ a_{12}^2 + a_{22}^2 &= 1 \rightarrow a_{12} = -a_{22} = \frac{1}{\sqrt{2}} \end{aligned}$$

The transformation between our small displacements and normal coordinates are then

$$\eta_i(t) = \sum_j a_{ij}\eta_j(t) \Rightarrow \vec{\boldsymbol{\eta}}(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\vec{\boldsymbol{\eta}}(t).$$

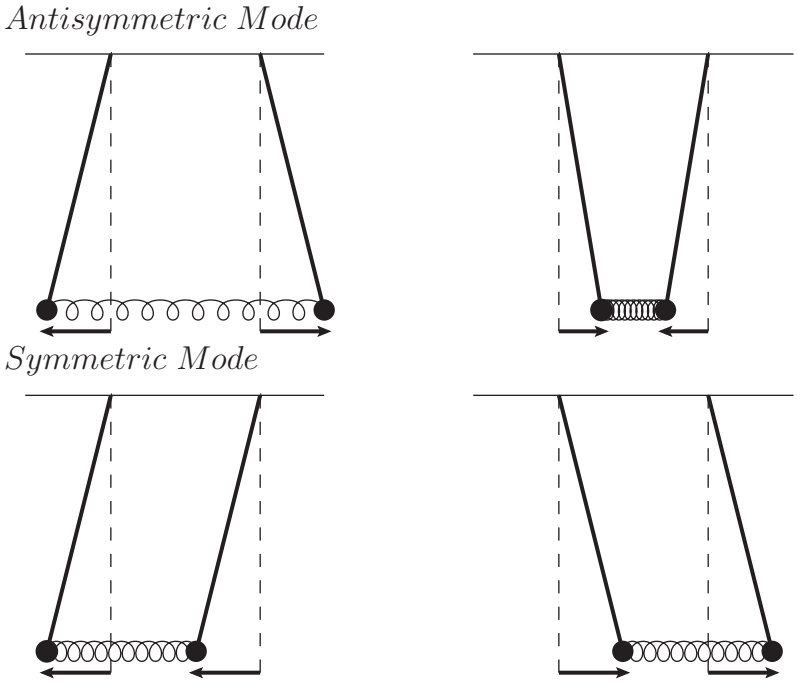
It turns out that this matrix is it's own inverse, so the normal coordinates are given by

$$\vec{\boldsymbol{\zeta}}(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\vec{\boldsymbol{\eta}}(t) \Rightarrow \begin{array}{l} \zeta_1 = \frac{1}{\sqrt{2}}(\eta_1 + \eta_2) \\ \zeta_2 = \frac{1}{\sqrt{2}}(\eta_1 - \eta_2) \end{array}.$$

Now to see what the physical meaning of these modes are, simply set the opposite one to zero. When $\zeta_1 = 0$, we have that $\eta_1 = -\eta_2$, and we have that ζ_2 corresponds to the antisymmetric mode. When $\zeta_2 = 0$, we have $\eta_1 = \eta_2$ and ζ_1 is therefore the symmetric phase (see figure).

2. Choose coordinates as in the figure. Note here that x_i are the small displacements. If the equilibrium length of the springs is l then we can put our coordinate system with $x = 0$ on the mass ν, so that the equilibrium positions of the three masses are $(-l, 0, l)$. The Lagrangian for this system is simply

$$L = \frac{1}{2}\mu(\dot{x}_1^2 + \dot{x}_3^2) + \frac{1}{2}\nu\dot{x}_2^2 - \frac{1}{2}k[(x_1 - x_2)^2 + (x_2 - x_3)^2].$$



(2.a)

In these coordinates, the equations of motion are

$$\sum_j (T_{ij}\ddot{x}_j + V_{ij}x_j) = 0,$$

where the matrices are

$$\mathbf{T} = \begin{pmatrix} \mu & & \\ & \nu & \\ & & \mu \end{pmatrix}, \quad \mathbf{V} = k \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Using the ansatz

$$x_j(t) = a_j e^{i\omega t}$$

we get the eigenequation

$$(\mathbf{V} - \mathbf{T}\omega^2)\vec{\boldsymbol{a}} = 0,$$

which has nontrivial solutions if and only if the determinant vanishes:

$$\begin{aligned} |\mathbf{V} - \mathbf{T}\omega^2| &= \begin{vmatrix} k-\mu\omega^2 & -k & 0 \\ -k & 2k-\nu\omega^2 & -k \\ 0 & -k & k-\mu\omega^2 \end{vmatrix} = 0 \\ &= \omega^2(k-\mu\omega^2)(\mu\nu\omega^2 - 2k\mu - k\nu) = 0. \end{aligned}$$

The solutions to this characteristic polynomial are

$$\omega_1 = \sqrt{\frac{k}{\mu}}, \quad \omega_2 = \sqrt{\frac{k}{\mu}\left(1 + 2\frac{\mu}{\nu}\right)}, \quad \omega_3 = 0.$$

These are the normal frequencies. To find the normal modes, generalize our solution to include a sum of these normal modes $x_j(t) = \sum_k a_{jk} e^{i\omega_k t}$, and so we solve

$$\sum_j (V_{ij} - T_{ij}\omega_k) a_{jk} = 0 \rightarrow \begin{pmatrix} (k-\omega_k\mu)a_{1k} & -ka_{2k} & 0 \\ -ka_{1k} & (2k-\omega_k^2\nu)a_{2k} & -ka_{3k} \\ 0 & -ka_{2k} & (k-\omega_k^2\mu)a_{3k} \end{pmatrix} = 0 \qquad (3)$$

for $k = 1, 2, 3$. First, setting $k = 1$ we see from the first line of (3) that $a_{21} = 0$, and from the second line that $a_{11} = -a_{31}$. Thus our eigenvector is

$$a_1 = a_{11} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad (4)$$

The constant a_{11} is free, and could be fixed with normalization conditions. For $k = 2$ we first see from the second line of (3) that $a_{22} = -2\frac{\mu}{\nu} a_{32}$, and plugging this into the first line we also see that $a_{12} = a_{32}$. Thus our second eigenvector is

$$a_2 = a_{32} \begin{pmatrix} 1 \\ -2\frac{\mu}{\nu} \\ 1 \end{pmatrix} \qquad (5)$$

With $k = 3$ in (3) we see immediately that $a_{13} = a_{23}$ and $a_{23} = a_{33}$ so we find

$$a_3 = a_{13} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad (6)$$

Applying the normalization condition

$$\sum_{i,j} T_{ij} a_{ir} a_{js} = \delta_{rs} \ ,$$

we find

$$a_{11} = = \frac{1}{\sqrt{2\mu}} \ , \qquad (7)$$

$$a_{32} = = \frac{1}{\sqrt{2\mu\left(1 + \frac{2\mu}{\nu}\right)}}$$

$$a_{13} = \frac{1}{\sqrt{2\mu + \nu}} \ . \qquad (8)$$

Now we can simply invert the equation

$$x_j = \sum_i a_{ij} \zeta_j$$

to find the normal modes ζ_j. The solution is

$$\zeta_1 = \sqrt{\frac{\mu}{2}}(x_1 - x_3)$$

$$\zeta_2 = \sqrt{\frac{\mu\nu}{2(2\mu+\nu)}}[x_1 - 2x_2 + x_3]$$

$$\zeta_3 = \frac{1}{\sqrt{2\mu+\nu}}(\mu x_1 + \nu x_2 + \mu x_3).$$

In the first normal mode (see figure (a)), $x_1 = -x_3$, so the two outer masses vibrate out of phase by 180° and with equal amplitudes. The central mass remains fixed (with zero amplitude). In the second mode (figure (b)), $x_1 = x_3$, so the two outer masses vibrate in phase at frequency ω_2 and with equal amplitudes. Because $x_2 = -2x_1\mu/\nu$, the central mass vibrates out of phase by 180° at the same frequency and with $2\mu/\nu$ times the amplitude. In the first and second modes the center of mass remains stationary. In the third normal mode (figure (c)), $x_1 = x_2 = x_3$, so the system moves as a whole. The center of mass moves at some fixed velocity v. Clearly there is no force. This is only due to translation of the CM.

(2.b)

The full solution to this problem is

$$\vec{\boldsymbol{x}}(t) = \vec{\boldsymbol{a}}_1(C_1 e^{i\omega_1 t} + C_1^* e^{-i\omega_1 t}) + \vec{\boldsymbol{a}}_2(C_2 e^{i\omega_2 t} + C_2^* e^{-i\omega_2 t}) + \vec{\boldsymbol{a}}_3(C + v_0 t),$$

where we have required that the solution be real and the motion associated with the CM has the form $C + v_o t$, *ie* constant translations in time.

(2.c)

The initial conditions along with the general solution above lead to a set of linear equations:

$$\begin{aligned}
x_1(0) &= C_1 + C_1^* + C_2 + C_2^* + C = -A \\
x_2(0) &= -2\frac{\mu}{\nu}(C_2 + C_2^*) + C = A\frac{\mu}{\nu} \\
x_3(0) &= -(C_1 C_1^*) + C_2 + C_2^* + C = 0 \\
\dot{x}_1(0) &= i\omega_1(C_1 - C_1^*) + i\omega_2(C_2 - C_2^*) + v_0 = 0 \\
\dot{x}_2(0) &= -2\frac{\mu}{\nu} i\omega_2(C_2 - C_2^*) + v_0 = 0 \\
\dot{x}_3(0) &= -i\omega_1(C_1 - C_1^*) + i\omega(C_2 - C_2^*) + v_0 = 0.
\end{aligned}$$

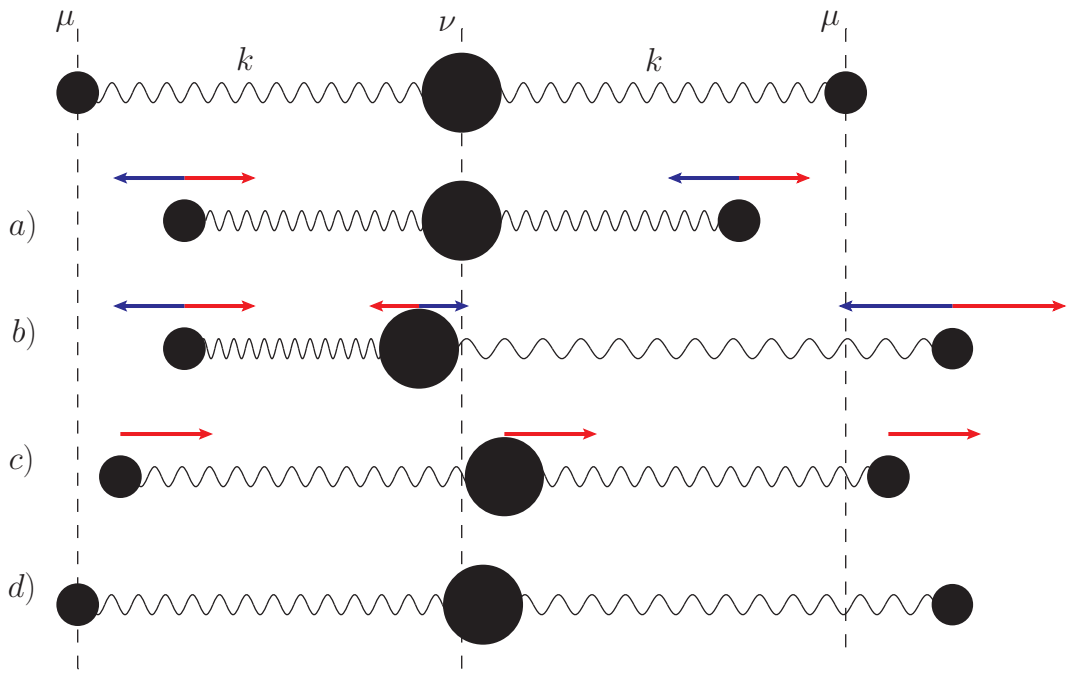
There are most easily solved with a matrix method,

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & -2\frac{\mu}{\nu} & -2\frac{\mu}{\nu} & 1 & 0 \\
-1 & -1 & 1 & 1 & 1 & 0 \\
\omega_1 & -\omega_1 & \omega_2 & -\omega_2 & 0 & -i \\
0 & 0 & -2\frac{\mu}{\nu}\omega_2 & 2\frac{\mu}{\nu}\omega_2 & 0 & -i \\
-\omega_1 & \omega_1 & \omega_2 & -\omega_2 & 0 & -i
\end{pmatrix}
\begin{pmatrix} C_1 \\ C_1^* \\ C_2 \\ C_2^* \\ C \\ v_0 \end{pmatrix}
=
\begin{pmatrix} -A \\ A\frac{\mu}{\nu} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Using standard reduction techniques one can see that the solution to this equation is

$$C = v_0 = 0,$$

$$C_1 = C_2 = -\frac{A}{4}.$$



Thus the solution is

$$\vec{\boldsymbol{x}}(t) = -\frac{A}{2}\begin{pmatrix} \cos\omega_1 t + \cos\omega_2 t \\ -2\frac{\mu}{\nu}\cos\omega_2 t \\ -\cos\omega_1 t + \cos\omega_2 t \end{pmatrix}$$

This situation is shown in figure (d). The CM remains fixed, the central mass performs simple harmonic motion at ω_2 and the other two masses move in a combination of the frequencies ω_1 and ω_2. If these two frequencies are close to each other, we have the phenomena of *beating*, although it is out of phase. In other words, when the amplitude of one of them is small, the amplitude of the other is large.