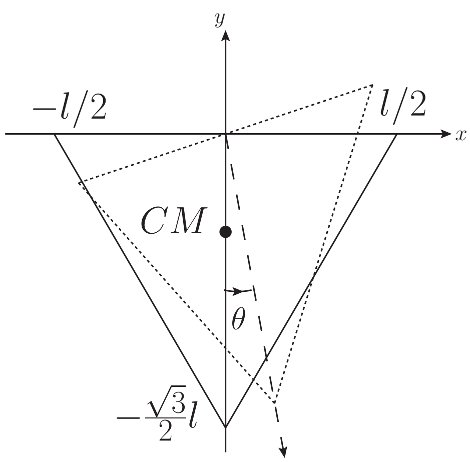
PHY 5246: Theoretical Dynamics, Fall 2015

Assignment # 9, Solutions

1 Graded Problems

Problem 1

(1.a)



In order to find the equation of motion of the triangle, we need to write the Lagrangian, with generalized coordinate $\{\theta\}$. The potential energy is going to be based on the location of the center of mass, and we find that via

$$
\begin{aligned}
y_{CM}(\theta=0) &= 2\frac{\rho}{m}\int_0^{l/2} dx \int_{-\sqrt{3}(l/2-x)}^{0} dy y \\
&= -\frac{2}{m}\frac{m}{\frac{1}{2}l\frac{\sqrt{3}}{2}l}\int_0^{l/2} dx \frac{1}{2}3\left(\frac{l}{2}-x\right)^2 \\
&= \frac{8}{\sqrt{3}l^2}\frac{3}{2}\frac{1}{3}\left(\frac{l}{2}-x\right)^2\Big|_0^{l/2} \\
&= -\frac{4}{\sqrt{3}l^2}\frac{l^3}{8} = -\frac{l}{2\sqrt{3}},
\end{aligned}
$$

where the density is ρ. This is as expected by symmetry considerations and by the fact that the CM has to be at the geometrical center of the triangle, *i.e.* at one third of it's height,

$$
-\frac{1}{3}\frac{\sqrt{3}}{2}l = -\frac{l}{2\sqrt{3}}.
$$

Note the negative sign is because the triangle lies below the x-axis. So, for generalized θ,

$$
y_{CM} = -\frac{\rho}{2\sqrt{3}}\cos\theta,
$$

and the potential energy is

$$
U = mgy_{CM} = -mg\frac{l}{2\sqrt{3}}\cos\theta.
$$

To calculate the kinetic energy,

$$
T = T_{rot}(\text{about } 0) = \frac{1}{2}I_3\dot{\theta}^2,
$$

we need first to find the principal moment of inertia about the axis of rotation, which is an axis perpendicular to the plane of the triangle, through 0. This is the only moment of inertia we need, since

$$\begin{aligned}
\vec{\omega} &= \dot{\theta}\hat{\mathbf{e}}_3 \text{ and therefore,} \\
T &= \frac{1}{2}\vec{\omega}^T \cdot \hat{\mathbf{I}} \cdot \vec{\omega} = \frac{1}{2}\dot{\theta}^2 I_{33} = \frac{1}{2}\dot{\theta}^2 I_3. \\
I_3 &= 2\rho \int_0^{l/2} \int_{-\sqrt{3}(l/2-x)}^{0} dy(x^2+y^2) \\
&= 2\rho \int_0^{l/2} dx x^2 \int_{-\sqrt{3}(l/2-x)}^{0} dy + 2\rho \int_0^{l/2} dx \int_{-\sqrt{3}(l/2-x)}^{0} dy y^2 \\
&= 2\rho \int_0^{l/2} dx x^2 \sqrt{3}\left(\frac{l}{2}-x\right) + 2\rho \int_0^{l/2} \frac{1}{3} 3\sqrt{3}\left(\frac{l}{2}-x\right)^3 \\
&= 2\rho\sqrt{3}\left(\frac{1}{3}\frac{l}{2}\frac{l^3}{8} - \frac{1}{3}\frac{l^4}{16} + \frac{1}{4}\frac{l^4}{16}\right) \\
&= 2\sqrt{3}\frac{m}{\frac{1}{2}\frac{\sqrt{3}}{2}l^2}\frac{1}{3}\frac{l^4}{16} = \frac{ml^2}{6}.
\end{aligned}$$

Therefore

$$T = \frac{1}{2}\left(\frac{ml^2}{6}\right)\dot{\theta}^2,$$

and the Lagrangian is

$$L = T - V = \frac{1}{2}\left(\frac{ml^2}{6}\right)\dot{\theta}^2 + mg\frac{l}{2\sqrt{3}}\cos\theta.$$

The equation of motion for our generalized coordinate is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \rightarrow \frac{ml^2}{6}\ddot{\theta} + mg\frac{l}{2\sqrt{3}}\sin\theta = 0$$

$$\Rightarrow \ddot{\theta} + \sqrt{3}\frac{g}{l}\sin\theta = 0.$$

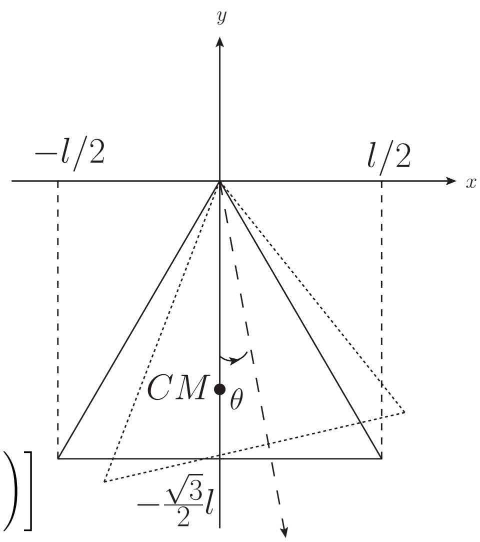
Under small oscillations about $\theta = 0$ this is the equation for a simple harmonic oscillator with frequency

$$\omega = \sqrt{\sqrt{3}\frac{g}{l}}.$$

(1.b)

For this situation, the $\theta = 0$ position of the CM is $y_{CM}(\theta = 0) = -\frac{l}{\sqrt{3}}$. Therefore

$$y_{CM} = -\frac{l}{\sqrt{3}}\cos\theta \Rightarrow U = -mg\frac{l}{\sqrt{3}}\cos\theta.$$



We need to calculate the moment of inertia I_3 with respect to the new center of rotation (and axis through it).

$$\begin{aligned}
I_3 &= 2\rho\int_0^{l/2} dx \int_{-\sqrt{3}/2l}^{-\sqrt{3}x} dx(x^2+y^2) \\
&= 2\rho\int_0^{l/2} dx x^2 \int_{-\frac{\sqrt{3}}{2}l}^{-\sqrt{3}x} dy + 2\rho\int_0^{l/2} dx \int_{-\frac{\sqrt{3}}{2}l}^{-\sqrt{3}x} dy y^2 \\
&= 2\rho\sqrt{3}\left[\int_0^{l/2} dx x^2\left(-x+\frac{l}{2}\right) + \int_0^{l/2} dx\frac{1}{3}3\left(-x^3+\frac{l^3}{8}\right)\right] \\
&= 2\rho\sqrt{3}\left(-\frac{1}{4}\frac{l^4}{16}+\frac{1}{3}l^4-\frac{1}{4}\frac{l^4}{16}+\frac{l^4}{16}\right) \\
&= 2\sqrt{3}\frac{m}{\frac{1}{2}\frac{\sqrt{3}}{2}l^2}\frac{l^4}{16}\frac{-3+2+6}{6} = \frac{5}{12}ml^2.
\end{aligned}$$

Thus the kinetic energy and Lagrangian are

$$T = \frac{1}{2}\left(\frac{5}{12}ml^2\right)\dot{\theta}^2$$

$$L = T - V = \frac{1}{2}\left(\frac{5}{12}ml^2\right)\dot{\theta}^2 + mg\frac{l}{\sqrt{3}}\cos\theta,$$

and the equation of motion is

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = 0 \to \frac{5}{12}ml^2\ddot{\theta} + mg\frac{l}{\sqrt{3}}\sin\theta = 0$$

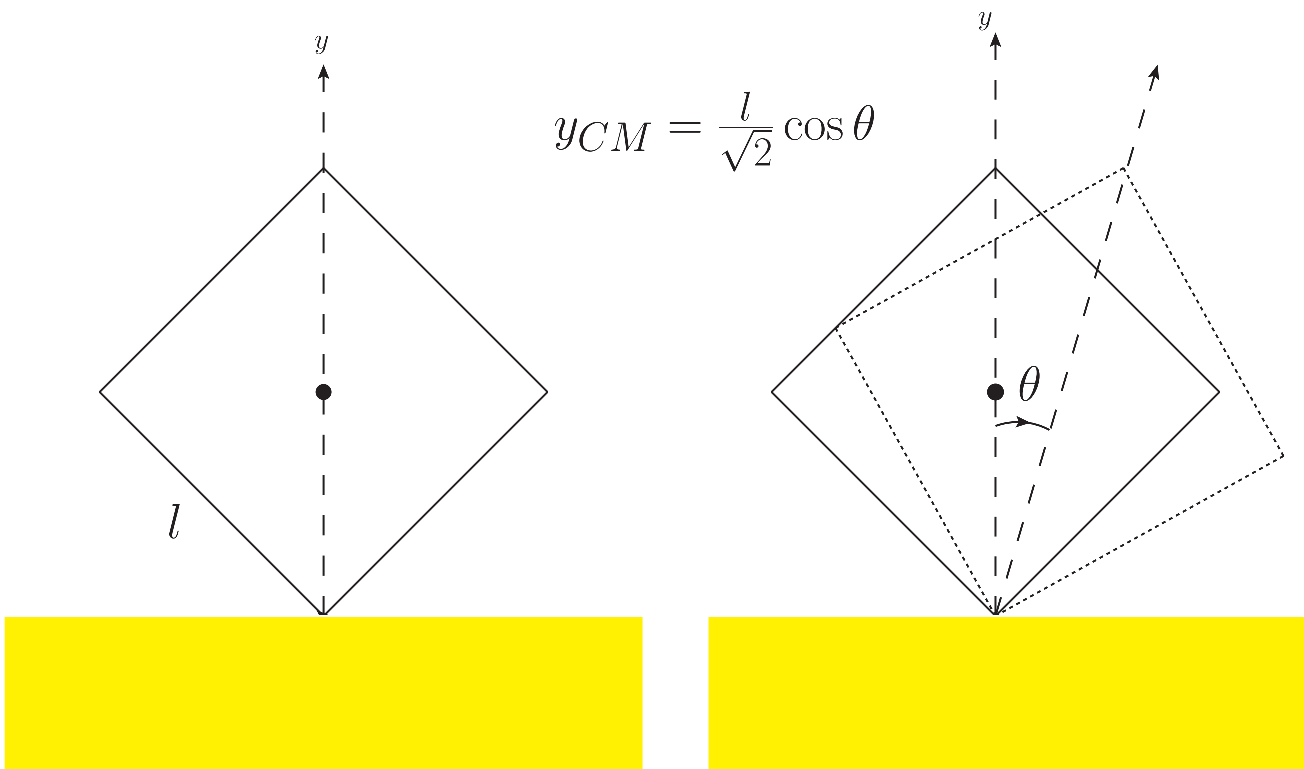
$$\Rightarrow \ddot{\theta} + \frac{g}{l}\frac{12}{5}\frac{1}{\sqrt{3}}\sin\theta = 0.$$

Thus the frequency of the small oscillation is

$$\omega = \sqrt{\frac{12}{5\sqrt{3}}\frac{g}{l}}.$$

Problem 2

(a)



The cube rotates about the side perpendicular to the plane of the figure, through 0. We are therefore referring to the case in which the frame has axes parallel to the side of the cube and origin at one corner. In figure (1), the cube is rotating about the back side, along the y-axis. The corresponding moment of inertia is $I_{22} = \frac{2}{3}ml^2$ (also see p. 409-410 in Goldstein). We can find the angular velocity of the cube when one face strikes the plane, by imposing conservation of energy:

$$\begin{aligned} E_i &= U_i = mg\frac{l}{\sqrt{2}} \\ E_f &= U_f + T_f = mg\frac{l}{2} + \frac{1}{2}\left(\frac{2}{3}ml^2\right)\omega_f^2 \\ E_i &= E_f \Rightarrow mg\frac{l}{\sqrt{2}} = mg\frac{l}{2} + \frac{1}{3}ml^2\omega_f^2. \end{aligned}$$

$$\Rightarrow g\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) = \frac{1}{3}l\omega_f^2 \Rightarrow \omega_f = \sqrt{\frac{3}{2}(\sqrt{2}-1)\frac{g}{l}}.$$

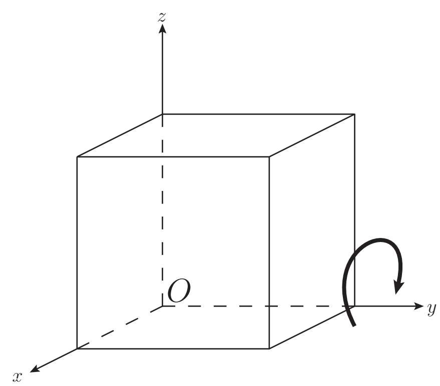


Figure 1:

(b)

If the cube can slide, there is no more fixed point. So, we have to calculate the kinetic energy as

$$\begin{aligned} T &= T_{trans}^{CM} + T_{rot}^{CM} \\ &= \frac{1}{2} m v_{CM}^2 + \frac{1}{2}\left(\frac{1}{6} m l^2\right)\omega_f^2, \end{aligned}$$

where:

$$v_{CM} = \dot{y}_{CM} = -\frac{l}{\sqrt{2}} \sin\theta\dot{\theta} = -\frac{l}{2}\dot{\theta},$$

and when $\theta = 45^{circ}$, the cube hits the plane. With $\omega(\theta = 45°) = \omega_f$, we have

$$T = \frac{1}{2} m \frac{l^2}{4}\omega_f^2 + \frac{1}{12} m l^2 \omega_f^2 = \frac{5}{24} l \omega_f^2,$$

Or

$$g\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) = \frac{5}{24} l\omega_f^2 \Rightarrow \omega_f = \sqrt{\frac{12}{5}(\sqrt{2}-1)\frac{g}{l}}.$$

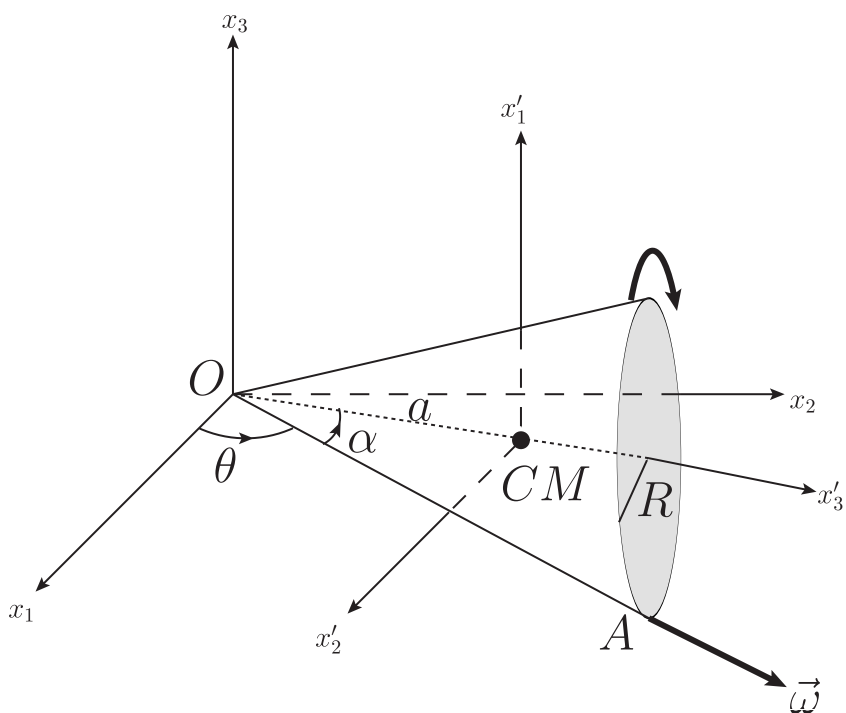
We could also have calculated the kinetic energy as

$$\begin{aligned} T &= T_{trans}^{CM} + T_{rot}^{CM} \\ &= \frac{1}{2} m v_{CM}^2 + \frac{1}{2} I_{22}^{CM}\omega_f^2 = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{6}\right) l^2 \omega_f^2 \\ &= \frac{1}{3} l^2 \omega_f^2 = T_{rot}^{origin}. \end{aligned}$$

Now use $v = R\dot{\theta}$ to find $v_{CM} = \frac{l}{\sqrt{2}}\omega_f$, and on p. 422 of Goldstein we can find $I_{22}^{CM} = \frac{ml^2}{6}$. We thus obtain the same answer as above.

Problem 3 (Goldstein 5.17)

Take M as the mass of the cone, (x'_1, x'_2, x'_3) as the principal axes going through the CM, which is at $(0, 0, a = \frac{3}{4}h)$ in the body system. x'_3 is the cone axis, x'_2 is perpendicular to the plane going through x'_3 and the segment OA, and x'_1 is orthogonal to the plane of x'_2 and x'_3. As stated in the problem, R is the radius, h is the height and α is the half-angle of the cone. Due to the condition of no-slipping, the angular velocity of the cone, $\vec{\omega}$, is directed along the instantaneous axis of rotation which is the line of contact between the surface of the cone and the surface on which the cone is rolling (i.e. along the segment OA, see figure).



(a)

The kinetic energy (T) can be calculated either using the center of mass as origin and taking as reference frame the system of principal axes described above, or using a system of axes parallel to that one but with origin at the apex of the cone O, which is a fixed point for the body.

In the first case, the kinetic energy is given by the sum of two terms: the kinetic energy *of the center of mass* (i.e. the kinetic energy of a pointlike object of mass M located at the CM of the cone and moving as CM moves) and the kinetic energy of the body as rotating about the center of mass (expressed in terms of the principal moments of inertia and the component of the angular velocity in the principal axes frame or body frame),

$$\begin{aligned} T &= T^{CM}_{trans} + T^{CM}_{rot} \\ T^{CM}_{trans} &= \begin{pmatrix} \text{kinetic energy} \\ \text{of the CM} \end{pmatrix} = \frac{1}{2} M v^2_{CM} = \frac{1}{2} M a^2 \cos^2 \alpha \dot{\theta}^2. \end{aligned}$$

In the above equation we have used that $v_{CM} = a\cos\alpha\dot{\theta}$ as it rotates about $\hat{\mathbf{x}}_3$. This implies that ω (angular velocity about $\overline{OA}$) and $\dot{\theta}$ are related. Thus we also have

$$v_{CM} = a\sin\alpha\omega.$$

Combining these two relations we find

$$\omega = \frac{v_{CM}}{a\sin\alpha} = \dot{\theta}\cot\alpha. \tag{1}$$

The components of $\vec{\omega}$ in the body frame are

$$\vec{\omega} = (\omega\sin\alpha, 0, \omega\cos\alpha) = (\dot{\theta}\cos\alpha, 0, \dot{\theta}\frac{\cos^2\alpha}{\sin\alpha}).$$

This follows from our choice of x'_2 and because of the relation (1). Therefore:

$$\begin{aligned}
T^{CM}_{rot} &= \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_3\omega_3^2 \\
&= \begin{cases} I_1 = I_2 &= \frac{3}{20}M\left(R^2 + \frac{h^2}{4}\right) \\ I_3 &= \frac{3}{10}MR^2. \end{cases} \\
&= \frac{1}{2}\frac{3}{20}M\left(R^2 + \frac{h^2}{4}\right)\cos^2\alpha\dot{\theta}^2 + \frac{1}{2}\frac{3}{10}MR^2\frac{\cos^4\alpha}{\sin^2\alpha}\dot{\theta}^2 \\
T &= T^{CM}_{trans} + T^{CM}_{rot} = \frac{1}{2}Ma^2\cos^2\alpha\dot{\theta}^2 + \frac{1}{2}\frac{3}{20}M\left(R^2 + \frac{h^2}{4}\right)\cos^2\alpha\dot{\theta}^2 + \frac{1}{2}\frac{3}{10}MR^2\frac{\cos^4\alpha}{\sin^2\alpha}\dot{\theta}^2 \\
&= \frac{9}{32}Mh^2\cos^2\alpha\dot{\theta}^2 + \frac{3}{40}Mh^2\left(\tan^2\alpha + \frac{1}{4}\right)\cos^2\alpha\dot{\theta}^2 + \frac{3}{20}Mh^2\cos^2\alpha\dot{\theta}^2.
\end{aligned}$$

To get this last line we used $a = \frac{3}{4}h$, $R = h\tan\alpha$, and the next line uses the trigonometric identity, $\tan^2\alpha = \frac{1}{\cos^2\alpha} - 1$.

$$\begin{aligned}
T &= \frac{9}{32}Mh^2\cos^2\alpha\dot{\theta}^2\frac{3}{40}Mh^2\dot{\theta}^2 - \frac{3}{40}\frac{3}{4}Mh^2\cos^2\alpha\dot{\theta}^2 + \frac{3}{20}Mh^2\cos^2\alpha\dot{\theta}^2 \\
&= \frac{3}{40}Mh^2\dot{\theta}^2(1 + 5\cos^2\alpha).
\end{aligned}$$

In the second case, the kinetic energy is purely rotational, since O is a fixed point, and is given by:

$$T = T^O_{rot} = \frac{1}{2}J_1\omega_1^2 + \frac{1}{2}J_3\omega_3^2\ ,$$

where J_1 and J_3 are the principal moments of inertia with respect to a system of axes parallel to the ones drawn in the picture and with origin in O. They can be found using Steiner's theorem and they are simply given by:

$$\begin{aligned}
J_1 &= I_1 + M(a^2 - a_1a_1) = I_1 + Ma^2\ , \\
J_3 &= I_3 + M(a^2 - a_3a_3) = I_3\ ,
\end{aligned}$$

where we have used that $\vec{a} = (0, 0, -a)$ is the position vector of O in the CM frame. It is then easy to see that,

$$T = T^O_{rot} = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}Ma^2\omega_1^2 + \frac{1}{2}I_3\omega_3^2 = \frac{1}{2}Ma^2\cos^2\alpha\dot{\theta}^2 + \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_3\omega_3^2 = T^{CM}_{trans} + T^{CM}_{rot}\ .$$

(b)

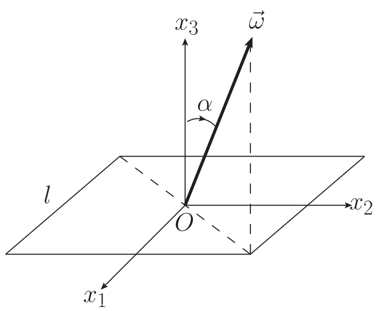
The components of the angular momentum are given by

$$\begin{aligned}
\vec{\mathbf{L}} &= L_1\hat{\mathbf{e}}'_1 + L_2\hat{\mathbf{e}}'_2 + L_3\hat{\mathbf{e}'}_3 \\
&= I_1\omega_1\hat{\mathbf{e}}'_1 + I_2\omega_2\hat{\mathbf{e}}'_2 + I_3\omega_3\hat{\mathbf{e}}'_3.
\end{aligned}$$

Therefore we know that $L_2 = 0$, and

$$\begin{aligned}
L_1 &= I_1\omega_1 = \frac{3}{20}M\left(R^2 + \frac{h^2}{4}\right)\cos\alpha\dot{\theta}^2 \\
&= \frac{3}{20}Mh^2\left(\frac{1}{\cos\alpha} - \frac{3}{4}\cos\alpha\right)\dot{\theta}, \\
L_3 &= I_3\omega_3 = \frac{3}{10}MR^2\frac{\cos^2\alpha}{\sin\alpha}\dot{\theta} = \frac{3}{10}Mh^2\tan^2\alpha\frac{\cos^2\alpha}{\sin\alpha}\dot{\theta} \\
&= \frac{3}{10}Mh^2\sin\alpha\dot{\theta}.
\end{aligned}$$

Problem 4



In this problem the body axes are the principal axes, and $\vec{\omega}$ can move in the the body fixed frame. It's easy to see that the plane is a symmetric top. Therefore, in absence of forces $\vec{\mathbf{L}}$ will be constant and $\vec{\omega}$ will precess around it.

Let us calculate the moments of inertia explicitly:

$$\begin{aligned} I_1 = I_2 &= \rho \int_{-l/2}^{l/2} dx \int_{-l/2}^{l/2} dy\ x^2 = \rho \frac{1}{3}\frac{2l^3}{8}\frac{l}{2}2 \\ &= \frac{ml^2}{12} \\ I_3 &= \rho \int_{-l/2}^{l/2} dx \int_{-l/2}^{l/2} dy\ (x^2+y^2) = \frac{ml^2}{6}. \end{aligned}$$

Now at $t=0$,

$$\vec{\omega} = \left(\frac{\omega \sin\alpha}{\sqrt{2}}, \frac{\omega \sin\alpha}{\sqrt{2}}, \omega\cos\alpha \right),$$

and the angular momentum is

$$\vec{\mathbf{L}} = (I_1\omega_1, I_2\omega_2, I_3\omega_3) = \frac{ml^2}{12}\left(\frac{\omega \sin\alpha}{\sqrt{2}}, \frac{\omega \sin\alpha}{\sqrt{2}}, 2\omega\sin\alpha \right).$$

The velocity with which $\vec{\omega}$ precesses about $\vec{\mathbf{L}}$ is (see discussion in class and in the text):

$$\Omega_{pr} = \frac{L}{I_1},$$

where

$$\begin{aligned} L &= (I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)^{1/2} = \frac{ml^2\omega}{6}\left[\frac{\sin^2\alpha}{8} + \frac{\sin^2\alpha}{8} + \cos^2\alpha \right]^{1/2} \\ &= \frac{ml^2\omega}{12}(1+3\cos^2\alpha)^{1/2}. \end{aligned}$$

And so the frequency of precession is

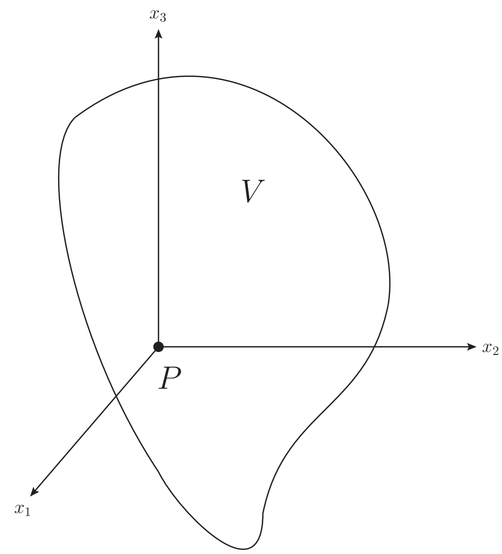
$$\Omega_{pr} = \frac{(ml^2\omega/12)(1+3\cos^2\alpha)^{1/2}}{ml^2/12} = \omega(1+3\cos^2\alpha)^{1/2}.$$

2 Non-graded Problems

Problem 5

Take a generic point P and the principal axis going through it (see figure). The principal moments of inertia are

$$\begin{aligned} I_1 &= \int_V d^3r f(\vec{\mathbf{r}}) \left[\vec{\mathbf{r}}^2 - x_1^2\right] \\ I_2 &= \int_V d^3r f(\vec{\mathbf{r}}) \left[\vec{\mathbf{r}}^2 - x_2^2\right] \\ I_3 &= \int_V d^3r f(\vec{\mathbf{r}}) \left[\vec{\mathbf{r}}^2 - x_3^2\right], \end{aligned}$$

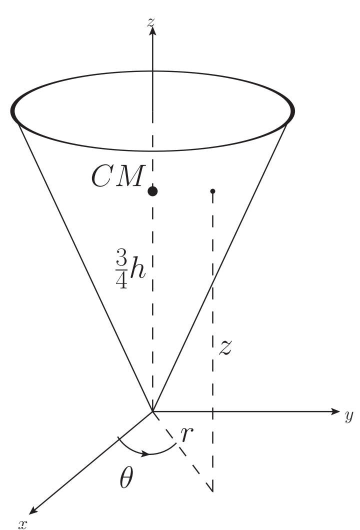


where $f(\vec{\mathbf{r}})$ is the mass density. Therefore

$$\begin{aligned} I_i + I_j &= \int_V d^3r f(\vec{\mathbf{r}}) \left[2\vec{\mathbf{r}}^2 - x_i^2 - x_j^2\right] \\ &= \int_V d^3r f(\vec{\mathbf{r}}) \left[x_i^2 + x_j^2 + 2x_k^2\right] \\ &\leq \int_V d^3r f(\vec{\mathbf{R}}) \left[x_i^2 + x_j^2\right] \\ &\leq \int_V d^3r f(\vec{\mathbf{r}}) \left[\vec{\mathbf{r}}^2 - x_k^2\right] = I_k. \end{aligned}$$

The equality in the above statement occurs when $x_k = 0$. Here we have assumed that $i \neq j \neq k$ for $i, j, k = 1, 2, 3$.

Problem 6



Using cylindrical coordinates,

$$\begin{cases} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z, \end{cases}$$

and we have that $r = (R/h)z$. First determine the inertia tensor with respect to 0:

$$\begin{aligned} I_{11} &= \rho \int_0^h dz \int_0^{2\pi} d\theta \int_0^{Rz/h} dr\ r(y^2 + z^2) \\ &= \frac{M}{\frac{1}{3}\pi R^2 h} \int_0^h dz \int_0^{2\pi} d\theta \int_0^{Rz/h} dr R^2 (r^2 \sin^2\theta + z^2) \\ &= \frac{M}{\frac{1}{3}\pi R^2 h} \left[\pi \int_0^h \frac{1}{4} \frac{R^4}{h^4} z^4 dz + 2\pi \int_0^h z^2 \frac{1}{2} \frac{R^2}{h^2} z^2 dz\right] \\ &= \frac{M}{\frac{1}{3}\pi R^2 h} R^2 h\pi \left[\frac{1}{20} R^2 + \frac{1}{5} h^2\right] = \frac{3}{20} M(R^2 + 4h^2) \\ I_{22} &= I_{11} \text{ by symmetry} \\ I_{33} &= \frac{M}{\frac{1}{3}\pi R^2 h} \int_0^h dz \int_0^{2\pi} d\theta \int_0^{Rz/h} dr\ r(x^2 + y^2) \\ &= \frac{M}{\frac{1}{2}\pi R^2 h} 2\pi \frac{R^4}{4h^4} \frac{h^5}{5} = \frac{3}{10} MR^2. \end{aligned}$$

By symmetry we can also say that the non diagonal terms are zero (each of the chosen axes is a symmetry axis for the cone). These can also be shown explicitly. For instance:

$$I_{12} = -\frac{M}{\frac{1}{3}\pi R^2 h}\int_0^h dz \int_0^{2\pi} d\theta \int_0^{Rz/h} dr\ rxy = 0.$$

This vanishes because $xy = r^2 \sin\theta\cos\theta$ and

$$\int_0^{2\pi} d\theta \sin\theta\cos\theta = 0.$$

Now in a similar way the other two vanish

$$\begin{aligned} I_{13} &= -\frac{M}{\frac{1}{3}\pi R^2 h}\int_0^h dz \int_0^{2\pi} d\theta \int_0^{Rz/h} dr\ rxz = 0, \\ I_{23} &= -\frac{M}{\frac{1}{3}\pi R^2 h}\int_0^h dz \int_0^{2\pi} d\theta \int_0^{Rz/h} dr\ ryz = 0 \end{aligned}$$

where we have used

$$\begin{aligned} xz = r\cos\theta z &\Rightarrow \int_0^{2\pi} d\theta \cos\theta = 0, \\ yz = r\sin\theta z &\Rightarrow \int_0^{2\pi} d\theta \sin\theta = 0. \end{aligned}$$

So, the chosen Cartesian axes are also principal axes with respect to the origin, and

$$\hat{I}_0 = \left\{ \begin{array}{ccc} \frac{3}{20}MR^2 & 0 & 0 \\ 0 & \frac{3}{20}MR^2 & 0 \\ 0 & 0 & \frac{3}{10}MR^2 \end{array} \right\}$$

To obtain $\hat{I}_{CM}$, we need first to determine (x_{CM}, y_{CM}, z_{CM}). By symmetry, we know $x_{CM} = y_{CM} = 0$, while:

$$\begin{aligned} I_{CM} &= \frac{1}{M}\frac{M}{\frac{1}{3}\pi R^2 h}\int_0^h dz \int_0^{2\pi} \int_0^{Rz/h} dr\ rz \\ &= \frac{3}{\pi R^2 h} 2\pi \int_0^h dz\ z \frac{1}{2}\frac{R^2}{h^2} z^2 \\ &= \frac{3}{\pi R^2 h} 2\pi \frac{1}{2}\frac{R^2}{h^2}\frac{h^4}{4} = \frac{3}{4}h. \end{aligned}$$

Finally, we can apply the generalized form of Steiner's parallel axis theorem, according to which

$$(I_{CM})_{ij} = (I_0)_{ij} - M\left[a^2\delta_{ij} - a_i a_j\right],$$

which in our case $\vec{\mathbf{a}} = (0, 0, \frac{3}{4}h)$, so we have

$$\begin{aligned} (I_{CM})_{11} &= (I_0)_{11} - \left(\frac{3}{4}h\right)^2 M = \frac{3}{20}M(R^2 + 4h^2) - \frac{9}{16}h^2 M \\ &= \frac{3}{20}M\left(R^2 \frac{h^2}{4}\right) \\ (I_{CM})_{22} &= (I_{CM})_{11} \text{ by symmetry} \\ (I_{CM})_{33} &= (I_0)_{33} = \frac{3}{10}MR^2, \end{aligned}$$

and the non diagonal elements are still all zero.

Problem 7

Defining ρ as the mass density, we use polar coordinates

$$\begin{cases} x &= r\cos\theta \\ y &= r\sin\theta \end{cases}$$

Given a coordinate system (x', y') which rotates with the disk (see figure (3)), the location of the CM is $\bar{x}_{CM} = 0$ and

$$\begin{aligned} \bar{y}_{CM} &= \frac{\rho}{M}\left\{\int_{top} dxdyy + 2\int_{bottom} dxdyy\right\} \\ &= \frac{\rho}{M}\left\{\int_0^R drr\int_0^\pi d\theta r\sin\theta + 2\int_0^R drr\int_\pi^{2\pi} d\theta r\sin\theta\right\} \\ &= \frac{\rho}{M}\left\{\frac{R^3}{3}2 - \frac{R^3}{3}4\right\} = -\frac{2}{3}\frac{\rho R^3}{M} = -\frac{4R}{9\pi}. \end{aligned}$$

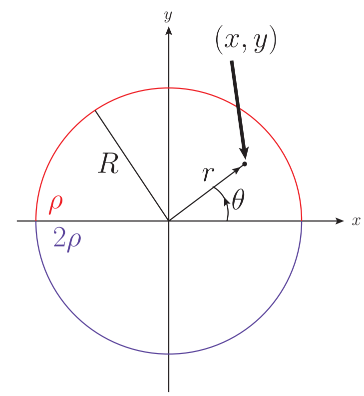


Figure 2:

Where in the last line we have used

$$M = \rho\frac{\pi R^2}{2} + 2\rho\frac{\pi R^2}{2} = \frac{3}{2}\rho\pi R^2.$$

Again referencing Figures (2) and (3) we see that the relationship between the lab coordinates and the coordinates of the center of mass are

$$\begin{cases} x_{CM} &= R\theta - |\bar{y}_{CM}|\sin\theta &= R\theta - \frac{4R}{9\pi}\sin\theta \\ y_{CM} &= R - |\bar{y}_{CM}|\cos\theta &= R - \frac{4R}{9\pi}\cos\theta. \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \dot{x}_{CM} &= R\dot{\theta} - \frac{4R}{9\pi}\cos\theta\dot{\theta} \\ \dot{y}_{CM} &= \frac{4R}{9\pi}\sin\theta\dot{\theta}. \end{cases}$$

The kinetic energy of the disk is made of 2 terms:

$$T = T_{trans} + T_{rot},$$

where T_{trans} is the translational kinetic energy of the mass M with coordinates (x_{CM}, y_{CM}) and velocity $(\dot{x}_{CM}, \dot{y}_{CM})$, and T_{rot} is the rotational kinetic energy about the center of mass. Thus

$$T = \frac{1}{2}M(\dot{x}_{CM}^2 + \dot{y}_{CM}^2) + \frac{1}{2}I_3\dot{\theta}^2.$$

Here I_3 is the moment of inertia about the CM, with respect to the z-axis, perpendicular to the plane of the figure. To find I_3 we will calculate it with respect to the center of disk (since it is easier!) and use Steiner's theorem to obtain it with respect to the CM.

$$\begin{aligned} I_3\Big|_0 &= \rho\int_0^R drr\int_0^\pi d\theta(x^2+y^2) + 2\rho\int_0^R drr\int_\pi^{2\pi} d\theta(x^2+y^2) \\ &= \rho\frac{R^4}{4}\pi + 2\rho\frac{R^4}{4}\pi = \rho R^4\frac{3}{4}\pi = \frac{1}{2}MR^2 \\ I_3\Big|_{CM} &= I_3\big|_0 - M\bar{y}_{CM}^2 = \frac{1}{2}MR^2 - M\frac{16}{81}\frac{R^2}{\pi^2} \\ &= \frac{1}{2}MR^2\left[1 - \frac{32}{81\pi^2}\right]. \end{aligned}$$

Then we can calculate the kinetic energy

$$\begin{aligned}
T &= T_{trans} + T_{rot} = \frac{1}{2}M(\dot{x}_{CM}^2 + \dot{y}_{CM}^2) + \frac{1}{2}I_3\Big|_{CM}\dot{\theta}^2 \\
&= \frac{1}{2}MR^2\dot{\theta}^2\left[\left(1 - \frac{4}{9\pi}\cos\theta\right)^2 + \left(\frac{4}{9\pi}\sin\theta\right)^2\right] + \frac{1}{4}MR^2\dot{\theta}^2\left[1 - \frac{32}{81\pi^2}\right] \\
&= \frac{1}{2}MR^2\dot{\theta}^2\left[1 + \frac{16}{81\pi^2} - \frac{8}{9\pi}\cos\theta + \frac{1}{2} - \frac{16}{81\pi^2}\right] \\
&= \frac{1}{2}MR^2\dot{\theta}^2\left[\frac{3}{2} - \frac{8}{9\pi}\cos\theta\right].
\end{aligned}$$

The potential energy must also be defined with respect to the center of mass,

$$U = Mgy_{CM} = MgR\left[1 - \frac{4}{9\pi}\cos\theta\right],$$

and the Lagrangian is the sum of these two terms,

$$L = T - V = \frac{1}{2}MR^2\dot{\theta}^2\left[\frac{3}{2} - \frac{8}{9\pi}\cos\theta\right] - mgR\left[1 - \frac{4}{9\pi}\cos\theta\right].$$

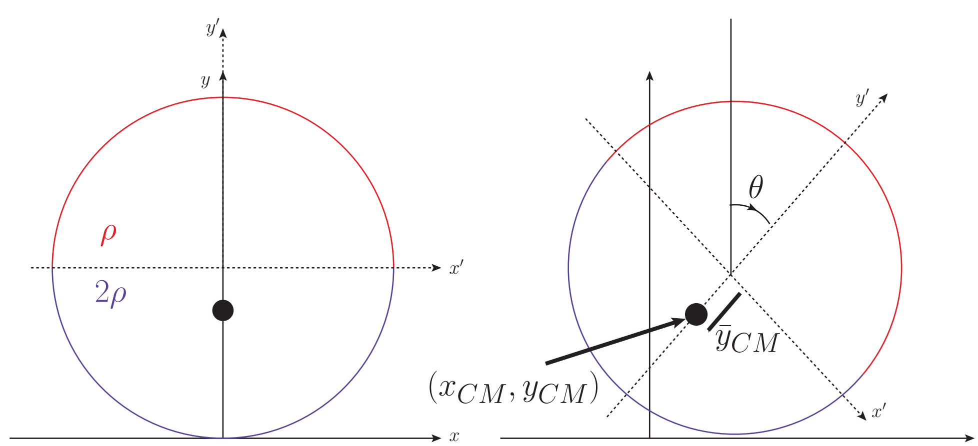


Figure 3: