PHY 5667 : Quantum Field Theory A, Fall 2015

September 3^{rd}, 2015
Assignment # 2
(due Thursday September 10^{th}, 2015)

1. Let us write the infinitesimal form of a Lorentz transformation in the *vector representation* as

$$\Lambda^\rho_\sigma = \delta^\rho_\sigma - \frac{i}{2}\delta\omega_{\mu\nu}(J^{\mu\nu}_V)^\rho_\sigma \ ,$$

where

$$(J^{\mu\nu}_V)^\rho_\sigma = i\,(g^{\mu\rho}\delta^\nu_\sigma - g^{\nu\rho}\delta^\mu_\sigma) \ ,$$

are matrices in the *vector representation* of the Lorentz generators ($g^{\mu\nu}$ denotes the metric tensor in Minkowski space and δ^ν_μ is the Kronecker δ in four dimensions).

1.a) Write Λ^μ_ν for a rotation by an angle θ about the x axis, and show that,

$$\Lambda = \exp\left(-i\theta J^{23}_V\right) \ .$$

1.b) Write Λ^μ_ν for a boost by rapidity η in the z direction, and show that,

$$\Lambda = \exp\left(i\eta J^{30}_V\right) \ .$$

2. Given the generators of the Lorentz algebra $J^{\mu\nu}$, defined by

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\,(g^{\mu\sigma}J^{\nu\rho} + g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho}) \ ,$$

2.a) define the generators of rotations and boosts as

$$L^i = \frac{1}{2}\epsilon^{ijk}J^{jk} \ , \quad K^i = J^{i0} \ \ (i,j = 1,2,3) \ ,$$

and show that an infinitesimal Lorentz transformation (acting on a generic object ϕ) can be written as:

$$\phi \to (1 - i\vec{\theta}\cdot\vec{L} + i\vec{\eta}\cdot\vec{K})\phi \ .$$

Write the commutation relations of these vector operators explicitly (for example: $[L_i, L_j] = \ldots$). Show that the combinations

$$\vec{J}_\pm = \frac{1}{2}\left(\vec{L} \pm i\vec{K}\right)$$

commute with one another and separately satisfy the commutation relations of angular momentum (i.e. of SU(2)).

2.b) Explain why the result of part **(2.a)** implies that all finite-dimensional representations of the Lorentz group correspond to pairs of integers or half-integers (j_-, j_+), corresponding to pairs of representations of the rotation group. Using the fact that $\vec{J} = \vec{\sigma}/2$ in the spin-1/2 representation of the angular momentum, write explicitly the transformation laws of the 2-component objects transforming according to the $\left(\frac{1}{2}, 0\right)$ (call it ψ_L) and $\left(0, \frac{1}{2}\right)$ (call it ψ_R) representations of the Lorentz group.

2.c) Using the identity $\vec{\sigma}^T = -\sigma^2 \vec{\sigma} \sigma^2$ show that an object that transforms according to the $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation can be represented as a (2×2) matrix that has the ψ_R transformation law on the left and the transposed ψ_L transformation law on the right. Parametrize this matrix as

$$\begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix} ,$$

and show that the object V^μ is a 4-vector.