PHY 5667 : Quantum Field Theory A, Fall 2015

September 10^{th}, 2015
Assignment # 3
(due Thursday September 24^{th}, 2015)

1. Consider a system of fields $\phi_i(x)$ and a Lorentz transformation acting on both field and coordinates as follows,

$$
\begin{aligned}
x'^{\mu} &= \Lambda^{\mu}_{\nu} x^{\nu} \simeq x^{\mu} + \sum_{k=1}^{6} \alpha_k X^{\mu}_k + O(\alpha^2) \ , \\
\phi'_i(x') &= L_{ij}(\Lambda)\phi_i(x) \simeq \phi_i(x) + \sum_{k=1}^{6} \alpha_k A_{ij,k}\phi_i(x) + O(\alpha^2) \ ,
\end{aligned}
$$

where $L(\Lambda)$ denote the representation of the Lorentz group on the space of the fileds $\phi_i(x)$. Show that if the action S,

$$
S = \int d^4x \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x), x) \ ,
$$

associated to the system of fields is invariant under such transformation, there are six conserved currents (Noether's currents) of the form,

$$
M^{\mu}_{\rho\sigma} = T^{\mu}_{\rho} x_{\sigma} - T^{\mu}_{\sigma} x_{\rho} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i(x))} A_{ij,\rho\sigma}\phi_j(x) \ ,
$$

where $T^{\mu\nu}$ is the energy-momentum tensor associated to the system of fields, and six conserved charges. How are the charges defined? How can you interpret the components of $M^{\mu}_{\rho\sigma}$ due to the transformation of the coordinates and to the trasformation of the fields respectively? Explain your reasoning.

2. Classical electromagnetism (with no sources) follows from the action,

$$
S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \ , \quad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ ,
$$

as you have derived in your first homework. Construct the energy-momentum tensor for this theory. Notice that the usual procedure does not result in a symmetric tensor. Show that you can remedy that using a tensor $K^{\lambda\mu\nu}$ antisymmetric in its first two indices (you will have to build a new energy-momentum tensor $\hat{T}^{\mu\nu} = T^{\mu\nu} + \ldots$ where the dots stays for a function of $K^{\lambda\mu\nu}$). In particular show that

$$
K^{\lambda\mu\nu} = F^{\mu\lambda} A^{\nu} \ ,
$$

leads to the standard formulae for the electromagnetic energy and momentum densities:

$$
\mathcal{E} = \frac{1}{2}(E^2 + B^2) \ , \qquad \mathbf{S} = \mathbf{E} \times \mathbf{B} \ .
$$

3. Recall that $T(a)^{-1}\phi(x)T(a) = \phi(x-a)$, where $T(a)\exp(iP^\mu a_\mu)$ is the spacetime translation operator, and P^0 is identified as the hamiltonian H.

3.a) Let a^μ be infinitesimal, and derive an expression for $[\phi(x), P^\mu]$.

3.b) Show that the time component of your result is equivalent to the Heisenberg equation of $i\dot{\phi} = [\phi, H]$.

3.c) For a free field, use the Heisenberg equation to derive the Klein- Gordon equation.

3.d) Define a spatial momentum operator

$$\mathbf{P} = -\int d^3x \Pi(x)\nabla\phi(x) \ ,$$

Use the canonical commutation relations to show that P obeys the relation you derived in part **3.a)**.

3.e) Express $\mathbf{P}$ in terms of $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$.

4. Problem 2.2 of Peskin and Schroeder's book.