Main Reference { your book
Bogoliubov & Shirkov, "Introduction to the theory of Quantized Fields" ("Noether's theorem and dynamics invariants")

(Global)
Symmetries and Conservation laws : Noether's theorem

"If the action functional (S) of a classical system of fields $q_i(x)$ is invariant under the action of a continuous group of transformations dependent on a finite number of parameters n, then the system admits n dynamical invariants, i.e. n conserved quantities (over time)".

$\hookrightarrow$ the group of transformations acts on both (or can act) coordinates and fields simultaneously.

Infinitesimal transformation:

$$x^\mu \rightarrow x'^\mu = x^\mu + \delta x^\mu \cong x^\mu + \sum_{k=1}^{n} X^\mu_{(k)} \alpha_k$$

$$q_i(x) \rightarrow q'_i(x') = q_i(x) + \delta q_i \cong q_i(x) + \sum_{k=1}^{n} \Phi_{i(k)} \alpha_k$$

$\{\alpha_k\} \rightarrow$ parameters

under this transformation:

$$\mathcal{L}(q_i(x), \partial_\mu q_i(x), x) \longrightarrow \mathcal{L}'(q'_i(x'), \partial_\mu q'_i(x'), x')$$

$$\mathcal{L}(x) \qquad\qquad \mathcal{L}'(x')$$

we want to compute:

$$\delta S = \frac{dS}{d\alpha} d\alpha = S(\alpha) - S(0) = \int d^4x' \mathcal{L}'(x') - \int d^4x \mathcal{L}(x)$$

$$\downarrow$$

$$S(\alpha) = S(0) + \sum_{k=1}^{n} \alpha_k \left.\frac{dS}{d\alpha_k}\right|_{\alpha=0} + O(\alpha^2)$$

extremum: $= 0$

and see the consequences of setting : $\delta S = 0$

First of all:

$$\mathcal{L}'(x') = \mathcal{L}(x) + \delta\mathcal{L}(x)$$

→ let's separate the variation due to δx^μ from the variation due to δq_i

$$= \mathcal{L}(x) + \bar{\delta}\mathcal{L}(x) + \frac{d\mathcal{L}(x)}{dx^\mu}\delta x^\mu$$

$$\bar{\delta}\mathcal{L}(x) = \frac{\partial \mathcal{L}}{\partial q_i}\bar{\delta} q_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)}\bar{\delta}(\partial_\mu q_i)$$

$$\frac{\partial \mathcal{L}}{\partial q_i}\frac{\partial q_i}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial(\partial_\nu q_i)}\frac{\partial(\partial_\nu q_i)}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial x^\mu}$$

$\bar{\delta} \rightarrow$ variation due only to change of q_i not due to δx^μ. (variation in form of q_i, not variation in its argument) $\Longrightarrow$

$$\begin{cases} \bar{\delta} q_i = q_i'(x) - q_i(x) = \delta q_i - \partial_\nu q_i \delta x^\nu \\ \bar{\delta}(\partial^\mu q_i) = \partial^\mu(\bar{\delta} q_i) \end{cases}$$

$$\bar{\delta}\mathcal{L}(x) = \frac{\partial \mathcal{L}}{\partial q_i}\bar{\delta} q_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)}\partial_\mu(\bar{\delta} q_i)$$

$$= \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)}\bar{\delta} q_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)}\partial_\mu(\bar{\delta} q_i)$$

$$= \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)}\bar{\delta} q_i\right)$$

So:

$$\mathcal{L}'(x') = \mathcal{L}(x) + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)}\bar{\delta} q_i\right) + \frac{d\mathcal{L}(x)}{dx^\mu}\delta x^\mu$$

Moreover:

$$d^4x' = dx_0' dx_1' dx_2' dx_3' = |J|\, d^4x =$$

$$= \det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) d^4x \simeq \left(1 + \frac{\partial \delta x^\mu}{\partial x^\mu}\right) d^4x$$

$$\det(1+\epsilon) = 1 + \mathrm{Tr}(\epsilon)$$

So :

$$\delta S' = \int d^4x' \mathcal{L}'(x') - \int d^4x \, \mathcal{L}(x)$$

$$= \int d^4x \left[\mathcal{L}(x) + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)} \bar{\delta} q_i \right) + \frac{d\mathcal{L}(x)}{dx^\mu} \delta x^\mu \right] \left(1 + \frac{\partial \delta x^{\mu'}}{\partial x^{\mu'}} \right)$$

$$- \int d^4x \, \mathcal{L}(x)$$

$$= \int d^4x \left[\cancel{\mathcal{L}(x)} + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)} \bar{\delta} q_i \right) + \frac{d\mathcal{L}(x)}{dx^\mu} \delta x^\mu + \mathcal{L}(x) \frac{\partial \delta x^\mu}{\partial x^\mu} - \cancel{\mathcal{L}(x)} \right]$$

$$= \int d^4x \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)} \bar{\delta} q_i \right) + \partial_\mu \left(\mathcal{L}(x) \delta x^\mu \right) \right]$$

we use ∂_μ as d_μ indeed from here on, since there are no more ambiguities.

$$= \int d^4x \, \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)} \bar{\delta} q_i + \mathcal{L}(x) \delta x^\mu \right]$$

$$= \int d^4x \, \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)} \left(\delta q_i - \partial_\nu q_i \, \delta x^\nu \right) + \mathcal{L}(x) \delta x^\mu \right]$$

$$= - \sum_{k=1}^{m} \int d^4x \, \partial_\mu \left[- \mathcal{L}(x) X^\mu_{(k)} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)} \left(\Phi_{i(k)} - \partial_\nu q_i \, X^\nu_{(k)} \right) \right] \alpha_k$$

$$= - \sum_{k=1}^{m} \int d^4x \, \partial_\mu \, J^\mu_{(k)}(x) \, \alpha_k$$

for later convenience (see Energy - Momentum tensor) where :

$$J^{\mu}_{(k)}(x) = - \mathcal{L}(x) X^{\mu}_{(k)} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)} \left(\Phi_{i(k)} - \partial_\nu q_i X^{\nu}_{(k)} \right)$$

so, finally :

$$\delta S = 0 \longrightarrow - \sum_{k=1}^{m} \int d^4x \, \partial_\mu J^{\mu}_{(k)}(x) \, \alpha_k = 0$$

since the $\{\alpha_k\}$ are independent :

$$\delta S = 0 \longrightarrow \frac{\partial S}{\partial \alpha_k} = 0 \longrightarrow \int d^4x \, \partial_\mu J^{\mu}_{(k)}(x) = 0$$

and because of the arbitrariness of the integration region this implies :

$$\partial_\mu J^{\mu}_{(k)}(x) = 0 \qquad k = 1, \dots, m$$

$\hookrightarrow$ <u>m conserved currents</u>

Moreover, for arbitrary space-like $\overset{3D}{\text{surfaces}}$ σ_1 and σ_2 we can rewrite :

$$\int d^4x \, \partial_\mu J^{\mu}_{(k)}(x) = \int_{\partial\Omega} dS_\mu \, J^{\mu}_{(k)}(x) = \int_{\sigma_1} dS_\mu \, J^{\mu}_{(k)} - \int_{\sigma_2} dS_\mu \, J^{\mu}_{(k)}$$

and

$$\int_{\sigma} dS_\mu \, J^{\mu}_{(k)}(x) = \text{constant}$$

choose $\sigma(t_0)$; $t_0 = \text{constant}$

$$C_k(t_0) = \int d^3x \, J^{0}_{(k)}(x) = \text{constant} \qquad k = 1, \dots, m$$

$\hookrightarrow$ <u>m conserved charges</u>, "constant of motion" or quantities

(1) space-time translations

$$x^\mu \to x'^\mu = x^\mu + a^\mu$$

infinitesimal: $x'^\mu = x^\mu + \epsilon^\mu$ $\qquad \delta x^\mu = \epsilon^\mu = \delta^\mu{}_\nu \epsilon^\nu$

$\downarrow$

$X^\mu{}_\nu \epsilon^\nu$

this is our X^μ_k $\qquad$ this is our ω_k's

$$\mathcal{J}^\mu_k \to \mathcal{J}^\mu{}_\nu = \mp T^\mu{}_{\nu'} X^{\nu'}{}_\nu = \pm T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta^\mu_\nu$$

talk about this arbitrariness

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \mathcal{L} g^{\mu\nu}$$

$\to$ energy-momentum tensor

$$Q_\nu = \int d^3\underline{x}\, \mathcal{J}^0{}_\nu(x) = \text{constant}$$

: conserved charges

$$= \int d^3\underline{x} \left[\frac{\partial \mathcal{L}}{\partial \dot\phi} \partial_\nu \phi - \mathcal{L} \delta^0_\nu \right]$$

$$Q_0 = \int d^3x \left[\frac{\partial \mathcal{L}}{\partial \dot\phi} \dot\phi - \mathcal{L} \right] = \int d^3x\, \mathcal{H} = H$$

energy of the system

$$Q_i = -\int d^3x \left[\frac{\partial \mathcal{L}}{\partial \dot\phi} \partial_i \phi \right] \equiv P_i$$

$\hookrightarrow$ momentum of the system

$\hookrightarrow$ see analogy with the classical case for a system with a discrete number of degrees of freedom.

the currents $J^\mu_{(k)}(x)$ are defined up to an arbitrary term of the form:

$$\partial_\nu f^{\mu\nu}_{(k)} \qquad \text{s.t.} \quad f^{\mu\nu}_{(k)} = -f^{\nu\mu}_{(k)}$$

since then:

$$\partial_\mu \partial_\nu f^{\mu\nu}_{(k)} = 0$$

(this goes back to the fact that the Lagrangian is defined up to a total 4-divergence).

<u>1st example</u> : Energy - momentum vector / tensor

infinitesimal space - time translations : $x'^\mu = x^\mu + a^\mu$

$$q_i'(x') = q_i(x) \qquad \alpha_k = a^\nu$$

such that: $\delta x^\mu = \sum_{k=1}^{n} X^\mu_{(k)} \alpha_k$ with $X^\mu_{(k)} = \delta^\mu_\nu$

$$\delta q_i = \sum_{k=1}^{n} \Phi_{i(k)} \alpha_k \quad \text{with } \Phi_{i(k)} = 0$$

the conserved currents are:

$$J^\mu_{\underset{\hookrightarrow \nu}{(k)}}(x) = -\mathcal{L}(x)\,\delta^\mu_\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)} \partial_\nu q_i$$

$$= -\mathcal{L}(x)\, g^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu q_i)} \partial^\nu q_i = T^{\mu\nu}$$

"energy momentum tensor"

and the conserved charges are:

$$P^\mu = \int d^3x\, T^{\mu 0}, \qquad \text{s.t.} \quad E = \int d^3x\, T^{00}$$

$$\left(= \frac{\partial \mathcal{L}}{\partial(\partial_0 q_i)} \partial^0 q_i - \mathcal{L} = H \right)$$

this is why we picked the overall $(-)$ sign.

and therefore the other three components are $\vec{P}$.