Lorentz and Poincaré symmetries in QFT

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We mentioned in the Introduction that quantum field theory (QFT) is a synthesis of the principles of quantum mechanics and of special relativity. Our first task will be to understand how Lorentz symmetry is implemented in field theory. We will study the representations of the Lorentz group in terms of fields and we will introduce scalar, spinor, and vector fields. We will then examine the information coming from Poincaré invariance. This chapter is rather mathematical and formal. The effort will pay, however, since an understanding of this group theoretical approach greatly simplifies the construction of the Lagrangians for the various fields in Chapter 3 and gives in general a deeper understanding of various aspects of QFT.

From now on we always use natural units $\hbar = c = 1$.

2.1 Lie groups

Lie groups play a central role in physics, and in this section we recall some of their main properties. In the next sections we will apply these concepts to the study of the Lorentz and Poincaré groups.

A Lie group is a group whose elements g depend in a continuous and differentiable way on a set of real parameters θ^a, $a = 1, \ldots, N$. Therefore a Lie group is at the same time a group and a differentiable manifold. We write a generic element as $g(\theta)$ and without loss of generality we choose the coordinates θ^a such that the identity element e of the group corresponds to $\theta^a = 0$, i.e. $g(0) = e$.

A (linear) *representation* R of a group is an operation that assigns to a generic, abstract element g of a group a linear operator $D_R(g)$ defined on a linear space,

$$g \mapsto D_R(g) \tag{2.1}$$

with the properties that

(i): $D_R(e) = 1$, where 1 is the identity operator, and

(ii): $D_R(g_1)D_R(g_2) = D_R(g_1 g_2)$, so that the mapping preserves the group structure.

The space on which the operators D_R act is called the *basis* for the representation R. A typical example of a representation is a *matrix representation*. In this case the basis is a vector space of finite dimension

n, and an abstract group element g is represented by a $n \times n$ matrix $(D_R(g))^i{}_j$, with $i, j = 1, \dots, n$. The *dimension* of the representation is defined as the dimension n of the base space. Writing a generic element of the base space as $(\phi^1, \dots, \phi^n)$, a group element g induces a transformation of the vector space

$$\phi^i \to (D_R(g))^i{}_j \phi^j \,. \tag{2.2}$$

Equation (2.2) allows us to attach a physical meaning to a group element: before introducing the concept of representation, a group element g is just an abstract mathematical object, defined by its composition rules with the other group members. Choosing a specific representation instead allows us to interpret g as a transformation on a certain space; for instance, taking as group $SO(3)$ and as base space the spatial vectors $\mathbf{v}$, an element $g \in SO(3)$ can be interpreted physically as a rotation in three-dimensional space.

A representation R is called reducible if it has an invariant subspace, i.e. if the action of any $D_R(g)$ on the vectors in the subspace gives another vector of the subspace. Conversely, a representation with no invariant subspace is called irreducible. A representation is *completely reducible* if, for all elements g, the matrices $D_R(g)$ can be written, with a suitable choice of basis, in block diagonal form. In other words, in a completely reducible representation the basis vectors ϕ^i can be chosen so that they split into subsets that do not mix with each other under eq. (2.2). This means that a completely reducible representation can be written, with a suitable choice of basis, as the direct sum of irreducible representations.

Two representations R, R' are called *equivalent* if there is a matrix S, independent of g, such that for all g we have $D_R(g) = S^{-1} D_{R'}(g) S$. Comparing with eq. (2.2), we see that equivalent representations correspond to a change of basis in the vector space spanned by the ϕ^i.

When we change the representation, in general the explicit form and even the dimensions of the matrices $D_R(g)$ will change. However, there is an important property of a Lie group that is independent of the representation. This is its *Lie algebra*, which we now introduce.

By the assumption of smoothness, for θ^a infinitesimal, i.e. in the neighborhood of the identity element, we have

$$D_R(\theta) \simeq 1 + i\theta_a T_R^a \,, \tag{2.3}$$

with

$$T_R^a \equiv -i \left. \frac{\partial D_R}{\partial \theta_a} \right|_{\theta=0} \,. \tag{2.4}$$

The T_R^a are called the *generators* of the group in the representation R. It can be shown that, with an appropriate choice of the parametrization far from the identity, the generic group elements $g(\theta)$ can always be represented by[1]

$$\boxed{D_R(g(\theta)) = e^{i\theta_a T_R^a} \,,} \tag{2.5}$$

[1] To be precise, this is only true for the component of the group manifold connected with the identity.

whose infinitesimal form reproduces eq. (2.3). The factor i in the definition (2.4) is chosen so that, if in the representation R the generators are hermitian, then the matrices $D_R(g)$ are unitary. In this case R is a *unitary representation.*

Given two matrices $D_R(g_1) = \exp(i\alpha_a T_R^a)$ and $D_R(g_2) = \exp(i\beta_a T_R^a)$, their product is equal to $D_R(g_1 g_2)$ and therefore must be of the form $\exp(i\delta_a T_R^a)$, for some $\delta_a(\alpha, \beta)$,

$$e^{i\alpha_a T_R^a} e^{i\beta_a T_R^a} = e^{i\delta_a T_R^a} . \tag{2.6}$$

Observe that T_R^a is a matrix. If A, B are matrices, in general $e^A e^B \neq e^{A+B}$, so in general $\delta_a \neq \alpha_a + \beta_a$. Taking the logarithm and expanding up to second order in α and β we get

$$\begin{aligned} i\delta_a T_R^a &= \log \left\{ [1 + i\alpha_a T_R^a + \frac{1}{2}(i\alpha_a T_R^a)^2][1 + i\beta_a T_R^a + \frac{1}{2}(i\beta_a T_R^a)^2] \right\} \qquad (2.7)\\ &= \log \left[1 + i(\alpha_a + \beta_a) T_R^a - \frac{1}{2}(\alpha_a T_R^a)^2 - \frac{1}{2}(\beta_a T_R^a)^2 - \alpha_a \beta_b T_R^a T_R^b \right] . \end{aligned}$$

Expanding the logarithm, $\log(1 + x) \simeq x - x^2/2$, and paying attention to the fact that the T_R^a do not commute we get

$$\alpha_a \beta_b \left[T_R^a, T_R^b \right] = i\gamma_c(\alpha, \beta) T_R^c , \tag{2.8}$$

with $\gamma_c(\alpha, \beta) = -2(\delta_c(\alpha, \beta) - \alpha_c - \beta_c)$. Since this must be true for all α and β, γ_c must be linear in α_a and in β_a, so the relation between γ and α, β must be of the general form $\gamma_c = \alpha_a \beta_b f^{ab}{}_c$ for some constants $f^{ab}{}_c$. Therefore

$$\boxed{[T^a, T^b] = i f^{ab}{}_c T^c .} \tag{2.9}$$

This is called the *Lie algebra* of the group under consideration. Two important points must be noted here. The first is that, even if the explicit form of the generators T^a depends on the representation used, the *structure constants* $f^{ab}{}_c$ are independent of the representation. In fact, if $f^{ab}{}_c$ were to depend on the representation, γ^a and therefore δ^a would also depend on R, so it would be of the form $\delta_R^a(\alpha, \beta)$. Then from eq. (2.6) we would conclude that the product of the group elements g_1 and g_2 gives a result which depends on the representation. This is impossible, since the result of the multiplication of two abstract group element $g_1 g_2$ is a property of the group, defined at the abstract group level without any reference to the representations. Therefore, we conclude that $f^{ab}{}_c$ are independent of the representation.[2] The second important point is that this equation has been derived requiring the consistency of eq. (2.6) to second order; however, once this is satisfied, it can be proved that no further requirement comes from the expansion at higher orders.

Thus the structure constants define the Lie algebra, and the problem of finding all matrix representations of a Lie algebra amounts to the algebraic problem of finding all possible matrix solutions T_R^a of eq. (2.9).

[2] Actually, the generators of a Lie group can even be defined without making any reference to a specific representation. One makes use of the fact that a Lie group is also a manifold, parametrized by the coordinates θ^a, and defines the generators as a basis of the tangent space at the origin. One then proves that their commutator (defined as a Lie bracket) is again a tangent vector, and therefore it must be a linear combination of the basis vector. In this approach no specific representation is ever mentioned, so it becomes obvious that the structure constants are independent of the representation. See, e.g., Nakahara (1990), Section 5.6.

A group is called *abelian* if all its elements commute between themselves, otherwise the group is *non-abelian*. For an abelian Lie group the structure constants vanish, since in this case in eq. (2.6) we have $\delta_a = \alpha_a + \beta_a$. The representation theory of abelian Lie algebras is very simple: any d-dimensional abelian Lie algebra is isomorphic to the direct sum of d one-dimensional abelian Lie algebras. In other words, all irreducible representations of abelian groups are one-dimensional. The non-trivial part of the representation theory of Lie algebras is related to the non-abelian structure.

In the study of the representations, an important role is played by the *Casimir operators*. These are operators constructed from the T^a that commute with all the T^a. In each irreducible representation, the Casimir operators are proportional to the identity matrix, and the proportionality constant labels the representation. For example, the angular momentum algebra is $[J^i, J^j] = i\epsilon^{ijk}J^k$ and the Casimir operator is $\mathbf{J}^2$. On an irreducible representation, $\mathbf{J}^2$ is equal to $j(j+1)$ times the identity matrix, with $j = 0, \frac{1}{2}, 1, \ldots$.

A Lie group that, considered as a manifold, is a compact manifold is called a compact group. Spatial rotations are an example of a compact Lie group, while we will see that the Lorentz group is non-compact. A theorem states that non-compact groups have no unitary representations of finite dimension, except for representations in which the non-compact generators are represented trivially, i.e. as zero. The physical relevance of this theorem is due to the fact that in a unitary representation the generators are hermitian operators and, according to the rules of quantum mechanics, only hermitian operators can be identified with observables. If a group is non-compact, in order to identify its generators with physical observables we need an infinite-dimensional representation. We will see in this chapter that the Lorentz and Poincaré groups are non-compact, and that infinite-dimensional representations are obtained introducing the Hilbert space of one-particle states.

2.2 The Lorentz group

The Lorentz group is defined as the group of linear coordinate transformations,

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{2.10}$$

which leave invariant the quantity

$$\eta_{\mu\nu}x^\mu x^\nu = t^2 - x^2 - y^2 - z^2 \,. \tag{2.11}$$

The group of transformations of a space with coordinates $(y_1, \ldots y_m, x_1, \ldots x_n)$, which leaves invariant the quadratic form $(y_1^2 + \ldots + y_m^2) - (x_1^2 + \ldots + x_n^2)$ is called the orthogonal group $O(n, m)$, so the Lorentz group is $O(3,1)$. The condition that the matrix Λ must satisfy in order to leave invariant the quadratic form (2.11) is

$$\eta_{\mu\nu}x'^\mu x'^\nu = \eta_{\mu\nu}(\Lambda^\mu{}_\rho x^\rho)(\Lambda^\nu{}_\sigma x^\sigma) = \eta_{\rho\sigma}x^\rho x^\sigma \,. \tag{2.12}$$

Since this must hold for x generic, we must have

$$\eta_{\rho\sigma} = \eta_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}\,. \tag{2.13}$$

In matrix notation, this can be rewritten as $\eta = \Lambda^T \eta \Lambda$. Taking the determinant of both sides, we therefore have $(\det\Lambda)^2 = 1$ or $\det\Lambda = \pm 1$. Transformations with $\det\Lambda = -1$ can always be written as the product of a transformation with $\det\Lambda = 1$ and of a discrete transformation that reverses the sign of an odd number of coordinates, e.g. a parity transformation $(t,x,y,z) \to (t,-x,-y,-z)$, or a reflection around a single spatial axis $(t,x,y,z) \to (t,-x,y,z)$, or a time-reversal transformation, $(t,x,y,z) \to (-t,x,y,z)$. Transformations with $\det\Lambda = +1$ are called *proper Lorentz transformations*. The subgroup of $O(3,1)$ with $\det\Lambda = 1$ is denoted by $SO(3,1)$.

Writing explicitly the 00 component of eq. (2.13) we find

$$1 = (\Lambda^0{}_0)^2 - \sum_{i=1}^{3}(\Lambda^i{}_0)^2 \tag{2.14}$$

which implies that $(\Lambda^0{}_0)^2 \geqslant 1$. Therefore the proper Lorentz group has two disconnected components, one with $\Lambda^0{}_0 \geqslant 1$ and one with $\Lambda^0{}_0 \leqslant -1$, called orthochronous and non-orthochronous, respectively. Any non-orthochronous transformation can be written as the product of an orthochronous transformation and a discrete inversion of the type $(t,x,y,z) \to (-t,-x,-y,-z)$, or $(t,x,y,z) \to (-t,-x,y,z)$, etc. It is convenient to factor out all these discrete transformations, and to redefine the Lorentz group as the component of $SO(3,1)$ for which $\Lambda^0{}_0 \geqslant 1$.

If we consider an infinitesimal transformation

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu} \tag{2.15}$$

eq. (2.13) gives

$$\omega_{\mu\nu} = -\omega_{\nu\mu}\,. \tag{2.16}$$

An antisymmetric 4×4 matrix has six independent elements, so the Lorentz group has six parameters. These are easily identified: first of all we have the transformations which leave t invariant. This is just the $SO(3)$ rotation group, generated by the three rotations in the (x,y), (x,z) and (y,z) planes. Furthermore, we have three transformations in the (t,x), (t,y) and (t,z) planes that leave invariant $t^2 - x^2$, etc. A transformation that leaves $t^2 - x^2$ invariant is called a *boost* along the x axis, and can be written as

$$t \to \gamma(t + vx)\,, \qquad x \to \gamma(x + vt)\,. \tag{2.17}$$

with $\gamma = (1-v^2)^{-1/2}$ and $-1 < v < 1$. Its physical meaning is understood looking at the small v limit, where it reduces to the velocity transformation of classical mechanics. It is therefore the relativistic generalization of a velocity transformation. The six independent parameters of the Lorentz group can therefore be taken as the three rotation angles and the three components of the velocity $\mathbf{v}$.

Since $-1 < v < 1$, we can write $v = \tanh\eta$, with $-\infty < \eta < +\infty$. Then $\gamma = \cosh\eta$ and eq. (2.17) can be written as a hyperbolic rotation,

$$\begin{aligned} t &\to (\cosh\eta)t + (\sinh\eta)x \\ x &\to (\sinh\eta)t + (\cosh\eta)x\,. \end{aligned} \tag{2.18}$$

The variable η is called the *rapidity*.

We see that the Lorentz group is parametrized in a continuous and differentiable way by six parameters, and it is therefore a Lie group. However, in the Lorentz group one of the parameters is the modulus of the boost velocity, $|\mathbf{v}|$, which ranges over the non-compact interval $0 \leqslant |\mathbf{v}| < 1$. Therefore the Lorentz group is non-compact.

2.3 The Lorentz algebra

We have seen that the Lorentz group has six parameters, the six independent elements of the antisymmetric matrix $\omega_{\mu\nu}$, to which correspond six generators. It is convenient to label the generators as $J^{\mu\nu}$, with a pair of antisymmetric indices (μ, ν), so that $J^{\mu\nu} = -J^{\nu\mu}$. A generic element Λ of the Lorentz group is therefore written as

$$\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}\,. \tag{2.19}$$

The factor 1/2 in the exponent compensates for the fact that we are summing over all μ, ν rather than over the independent pairs with $\mu < \nu$, and therefore each generator is counted twice.

By definition a set of objects ϕ^i, with $i = 1, \ldots, n$, transforms in a representation R of dimension n of the Lorentz group if, under a Lorentz transformation,

$$\phi^i \to \left[e^{-\frac{i}{2}\omega_{\mu\nu}J_R^{\mu\nu}}\right]^i{}_j\,\phi^j\,, \tag{2.20}$$

where $\exp\{-(i/2)\omega_{\mu\nu}J_R^{\mu\nu}\}$ is a matrix representation of dimension n of the abstract element (2.19) of the Lorentz group; $J_R^{\mu\nu}$ are the Lorentz generators in the representation R, and are $n \times n$ matrices. Under an infinitesimal transformation with infinitesimal parameters $\omega_{\mu\nu}$, the variation of ϕ^i is

$$\delta\phi^i = -\frac{i}{2}\omega_{\mu\nu}(J_R^{\mu\nu})^i{}_j\phi^j\,. \tag{2.21}$$

In $(J_R^{\mu\nu})^i{}_j$ the pair of indices μ, ν identify the generator while the indices i, j are the matrix indices of the representation that we are considering.

All physical quantities can be classified accordingly to their transformation properties under the Lorentz group. A scalar is a quantity that is invariant under the transformation. A typical Lorentz scalar in particle physics is the rest mass of a particle. A *contravariant four-vector* V^μ is defined as an object that satisfies the transformation law

$$V^\mu \to \Lambda^\mu{}_\nu V^\nu\,, \tag{2.22}$$

with $\Lambda^\mu{}_\nu$ defined by the condition (2.13). A covariant four-vector V_μ transforms as $V_\mu \to \Lambda_\mu{}^\nu V_\nu$, with $\Lambda_\mu{}^\nu = \eta_{\mu\rho}\eta^{\nu\sigma}\Lambda^\rho{}_\sigma$. One immediately verifies that, if V^μ is a contravariant four-vector, then $V_\mu \equiv \eta_{\mu\nu}V^\nu$ is a covariant four-vector. We refer generically to covariant and contravariant four-vectors simply as four-vectors. The space-time coordinates x^μ are the simplest example of four-vector. Another particularly important example is given by the four-momentum $p^\mu = (E, \mathbf{p})$.

The explicit form of the generators $(J_R^{\mu\nu})^i{}_j$ as $n \times n$ matrices depends on the particular representation that we are considering. For a scalar ϕ, the index i takes only one value, so it is a one-dimensional representation, and $(J^{\mu\nu})^i{}_j$ is a 1×1 matrix, i.e. a number, for each given pair (μ, ν). But in fact, by definition, on a scalar a Lorentz transformation is the identity transformation, so $\delta\phi = 0$ and $J^{\mu\nu} = 0$. A representation in which all generators are equal to zero is trivially a solution of eq. (2.9), for any Lie group, and so it is called the trivial representation.

The four-vector representation is more interesting. In this case i, j are themselves Lorentz indices, so each generator $J^{\mu\nu}$ is represented by a 4×4 matrix $(J^{\mu\nu})^\rho{}_\sigma$. The explicit form of this matrix is

$$(J^{\mu\nu})^\rho{}_\sigma = i\left(\eta^{\mu\rho}\,\delta^\nu_\sigma - \eta^{\nu\rho}\,\delta^\mu_\sigma\right) . \tag{2.23}$$

This can be shown observing that, from eqs. (2.22) and (2.15), the variation of a four-vector V^μ under an infinitesimal Lorentz transformation is $\delta V^\mu = \omega^\mu{}_\nu V^\nu$, which can be rewritten as

$$\delta V^\rho = -\frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^\rho{}_\sigma\, V^\sigma , \tag{2.24}$$

with $(J^{\mu\nu})^\rho{}_\sigma$ given by eq. (2.23) (this solution for $J^{\mu\nu}$ is unique because we require the antisymmetry under $\mu \leftrightarrow \nu$). This representation is irreducible since a generic Lorentz transformation mixes all four components of a four-vector and therefore there is no change of basis that allows us to write $(J^{\mu\nu})^\rho{}_\sigma$ in block diagonal form. We can now use the explicit expression (2.23) to compute the commutators, and we find

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}\right) . \tag{2.25}$$

This is the Lie algebra of $SO(3,1)$. It is convenient to rearrange the six components of $J^{\mu\nu}$ into two spatial vectors,

$$J^i = \frac{1}{2}\epsilon^{ijk}J^{jk} , \qquad K^i = J^{i0} . \tag{2.26}$$

In terms of J^i, K^i the Lie algebra of the Lorentz group (2.25) becomes

$$[J^i, J^j] = i\epsilon^{ijk}J^k , \tag{2.27}$$

$$[J^i, K^j] = i\epsilon^{ijk}K^k , \tag{2.28}$$

$$[K^i, K^j] = -i\epsilon^{ijk}J^k . \tag{2.29}$$

Equation (2.27) is the Lie algebra of $SU(2)$ and this shows that J^i, defined in eq. (2.26), is the angular momentum. Instead eq. (2.28) expresses the fact that $\mathbf{K}$ is a spatial vector.

We also introduce the definitions $\theta^i = (1/2)\epsilon^{ijk}\omega^{jk}$ and $\eta^i = \omega^{i0}$. Then

$$\begin{aligned}\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} &= \omega_{12}J^{12} + \omega_{13}J^{13} + \omega_{23}J^{23} + \sum_{i=1}^{3}\omega_{i0}J^{i0} \\ &= \boldsymbol{\theta}\cdot\mathbf{J} - \boldsymbol{\eta}\cdot\mathbf{K}\,,\end{aligned} \tag{2.30}$$

where we used $\omega_{i0} = -\omega^{i0} = -\eta^i$ while $\omega_{12} = \omega^{12} = \theta^3$, etc. Then a Lorentz transformation can be written as

$$\Lambda = \exp\{-i\boldsymbol{\theta}\cdot\mathbf{J} + i\boldsymbol{\eta}\cdot\mathbf{K}\}\,. \tag{2.31}$$

With our definitions $\theta^i = +(1/2)\epsilon^{ijk}\omega^{jk}$ and $\eta^i = +\omega^{i0}$ a rotation by an angle $\theta > 0$ in the (x, y) plane rotates *counterclockwise* the position of a point P with respect to a fixed reference frame,[3] while performing a boost of velocity $\mathbf{v}$ on a particle at rest we get a particle with velocity $+\mathbf{v}$. To check these signs, we can consider infinitesimal transformations, and use the explicit form (2.23) of the generators. Performing a rotation by an angle θ around the z axis, eqs. (2.31) and (2.23) give

$$\delta x^\mu = -i\theta(J^{12})^\mu{}_\nu x^\nu = \theta\,(\eta^{1\mu}\delta^2_\nu - \eta^{2\mu}\delta^1_\nu)x^\nu \tag{2.32}$$

and therefore $\delta x = -\theta y$ and $\delta y = +\theta x$, corresponding to a counterclockwise rotation. Similarly, performing a boost along the x axis,

$$\delta x^\mu = +i\eta(J^{10})^\mu{}_\nu x^\nu = -\eta\,(\eta^{1\mu}\delta^0_\nu - \eta^{0\mu}\delta^1_\nu)x^\nu \tag{2.33}$$

and therefore $\delta t = +\eta\, x$ and $\delta x = +\eta\, t$, which is the infinitesimal form of eq. (2.18).

2.4 Tensor representations

By definition a tensor $T^{\mu\nu}$ with two contravariant (i.e. upper) indices is an object that transforms as

$$T^{\mu\nu} \to \Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}T^{\mu'\nu'}\,. \tag{2.34}$$

In general, a tensor with an arbitrary number of upper and lower indices transforms with a factor $\Lambda^\mu{}_{\mu'}$ for each upper index and a factor $\Lambda_\mu{}^{\mu'}$ for each lower index.

Tensors are examples of representations of the Lorentz group. For instance, a generic tensor $T^{\mu\nu}$ with two indices has 16 components and eq. (2.34) shows that these 16 components transform among themselves, i.e. they are a basis for a representation of dimension 16. However, this representation is reducible. From eq. (2.34) we see that, if $T^{\mu\nu}$ is antisymmetric, after a Lorentz transformation it remains antisymmetric, while if it is symmetric it remains symmetric. So the symmetric and antisymmetric parts of a tensor $T^{\mu\nu}$ do not mix, and the 16-dimensional

[3] This is the "active" point of view. Alternatively, we can say that we keep P fixed and we rotate the reference frame clockwise; this is the "passive" point of view.

representation is reducible into a six-dimensional antisymmetric representation $A^{\mu\nu} = (1/2)(T^{\mu\nu} - T^{\nu\mu})$ and a 10-dimensional symmetric representation $S^{\mu\nu} = (1/2)(T^{\mu\nu} + T^{\nu\mu})$. Furthermore, also the trace of a symmetric tensor is invariant,

$$S \equiv \eta_{\mu\nu} S^{\mu\nu} \to \eta_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} S^{\rho\sigma} = S\,, \tag{2.35}$$

where in the last step we used the defining property of the Lorentz group, eq. (2.13). This means, in particular, that a traceless tensor remains traceless after a Lorentz transformation, and thus the 10-dimensional symmetric representation decomposes further into a nine-dimensional irreducible symmetric traceless representation, $S^{\mu\nu} - (1/4)\eta^{\mu\nu} S$, and the one-dimensional scalar representation S.

The following notation is commonly used: an irreducible representation is denoted by its dimensionality, written in boldface. Thus the scalar representation is denoted as **1**, the four-vector representation as **4**, the antisymmetric tensor as **6** and the traceless symmetric tensor as **9**.[4] The tensor representation (2.34) is a tensor product of two four-vector representations, which means that each of the two indices of $T^{\mu\nu}$ transforms separately as a four-vector index, i.e. with the matrix Λ. The tensor product of two representations is denoted by the symbol $\otimes$. We have found above that the tensor product of two four-vector representations decomposes into the direct sum of the **1**, **6**, and **9** representations. Denoting the direct sum by $\oplus$, we have[5]

$$\mathbf{4} \otimes \mathbf{4} = \mathbf{1} \oplus \mathbf{6} \oplus \mathbf{9}\,. \tag{2.36}$$

The decomposition into irreducible representations of tensors with more than two indices can be obtained similarly. The most general irreducible tensor representations of the Lorentz group are found starting from a generic tensor with an arbitrary number of indices, removing first all traces, and then symmetrizing or antisymmetrizing over all pairs of indices. Note that, using $\eta^{\mu\nu}$, we can always restrict to contravariant tensors; for instance V^{μ} and V_{μ} are equivalent representations.

All tensor representations are in a sense derived from the four-vector representation, since the transformation law of a tensor is obtained applying separately on each Lorentz index the matrix $\Lambda^{\mu}{}_{\nu}$ that defines the transformation of four-vectors. This means that (as the name suggests) tensor representations are tensor products of the four-vector representation. For this reason, the four-vector representation plays a distinguished role and is called the *fundamental* representation of $SO(3,1)$.[6]

Another representation of special importance is the *adjoint* representation. It is a representation which has the same dimension as the number of generators. This means that we can use the same type of indices a, b, c for labeling the generator and its matrix elements, and for any Lie group it can be written in full generality in terms of the structure constants, as

$$(T^{a}_{\text{adj}})^{b}{}_{c} = -i f^{ab}{}_{c}\,. \tag{2.37}$$

The Lie algebra (2.9) is automatically satisfied by (2.37). This follows from the fact that, for all matrices A, B, C, there is an algebraic identity

[4] If two inequivalent representations happen to have the same dimensionality one can use a prime or an index to distinguish between them.

[5] In Exercise 2.5 we discuss the separation of the representation **6**, i.e. the antisymmetric tensor, into its self-dual and anti-self-dual parts, both in Minkowski space and in a Euclidean space with metric $\delta^{\mu\nu}$. We will see that in the Euclidean case the antisymmetric tensor $A^{\mu\nu}$ is reducible and decomposes into two three-dimensional representations corresponding to self-dual and anti-self-dual tensors, while in Minkowski space an antisymmetric tensor $A^{\mu\nu}$ with real components is irreducible.

[6] To avoid all misunderstanding, we anticipate that in Section 2.5 we will enlarge the definition of the Lorentz group to include spinorial representations. With this enlarged definition, four-vectors are no longer the fundamental representation of the Lorentz group. Instead, all representations of the Lorentz group will be built from the spinorial representations $(1/2, 0)$ and $(0, 1/2)$ that will be defined in Section 2.5.

known as the *Jacobi identity*,

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0\,, \tag{2.38}$$

which is easily verified writing the commutators explicitly. Setting in this identity $A=T^a, B=T^b$ and $C=T^c$ we find that the structure constants of any Lie group obey the identity

$$f^{ab}{}_d f^{cd}{}_e + f^{bc}{}_d f^{ad}{}_e + f^{ca}{}_d f^{bd}{}_e = 0\,. \tag{2.39}$$

If we substitute eq. (2.37) into eq. (2.9), we see that the Lie algebra is automatically satisfied because of eq. (2.39).

For the Lorentz group, the adjoint representation has dimension six, so it is given by the antisymmetric tensor $A^{\mu\nu}$. The adjoint representation plays an especially important role in non-abelian gauge theories, as we will see in Chapter 10.

All the representation theory on tensors that we have developed having in mind $SO(3,1)$ goes through for $SO(n)$ or $SO(n,m)$ generic, simply replacing $\eta_{\mu\nu}$ with $\delta_{\mu\nu}$ for $SO(n)$, or with a diagonal matrix with n minus signs and m plus sign for $SO(n,m)$.

2.4.1 Decomposition of Lorentz tensors under $SO(3)$

Since we know how a tensor behaves under a generic Lorentz transformation, we know in particular its transformation properties under the $SO(3)$ rotation subgroup, and we can therefore ask what is the angular momentum j of the various tensor representations. Recall that the representations of $SO(3)$ are labeled by an index j which takes integer values $j=0,1,2,\ldots$, and the dimension of the representation labeled by j is $2j+1$. Within each representation, these $2j+1$ states are labeled by $j_z=-j,\ldots,j$. For $SO(3)$, it is more common to denote the representation as $\mathbf{j}$, i.e. to label it with the angular momentum rather than with the dimension of the representation, $2j+1$. In this notation, $\mathbf{0}$ is the scalar (also called the singlet), $\mathbf{1}$ is a triplet with components $j_z=-1,0,1$, while $\mathbf{2}$ is a representation of dimension 5, etc. (if we rather use the same convention as in the case of the Lorentz group, i.e. we label them by their dimensionality, we should write $\mathbf{1},\mathbf{3},\mathbf{5},\ldots$).

A Lorentz scalar is of course also scalar under rotations, so it has $j=0$. A four-vector $V^\mu=(V^0,\mathbf{V})$ is an irreducible representation of the Lorentz group, since a generic Lorentz transformation mixes all four components, but from the point of view of the $SO(3)$ subgroup it is reducible: spatial rotations do not mix V^0 with $\mathbf{V}$; V^0 is invariant under spatial rotations, so it has $j=0$, while the three spatial components V^i form an irreducible three-dimensional representation of $SO(3)$, so they have $j=1$. In group theory language we say that, from the point of view of spatial rotations, a four-vector decomposes into the direct sum of a scalar and a $j=1$ representation,

$$V^\mu \in \mathbf{0}\oplus\mathbf{1} \tag{2.40}$$

or, if we prefer to label the representations by their dimension, rather than by j, we write $\mathbf{4} = \mathbf{1} \oplus \mathbf{3}$. The former notation indicates more clearly what are the spins involved while the latter makes apparent that the number of degrees of freedom on the left-hand side matches those on the right-hand side.

We now want to understand what angular momenta appear in a generic tensor $T^{\mu\nu}$ with two indices. By definition a tensor $T^{\mu\nu}$ transforms as the tensor product of two four-vector representations. Since, from the point of view of $SO(3)$, a four-vector is $\mathbf{0} \oplus \mathbf{1}$, a generic tensor with two indices has the following decomposition in angular momenta

$$\begin{aligned} T^{\mu\nu} \in (\mathbf{0} \oplus \mathbf{1}) \otimes (\mathbf{0} \oplus \mathbf{1}) &= (\mathbf{0} \otimes \mathbf{0}) \oplus (\mathbf{0} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{0}) \oplus (\mathbf{1} \otimes \mathbf{1}) \\ &= \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{1} \oplus (\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}) \,. \end{aligned} \tag{2.41}$$

In the last step we used the usual rule of composition of angular momenta, which says that composing two angular momenta j_1 and j_2 we get all angular momenta between $|j_1 - j_2|$ and $j_1 + j_2$, so $\mathbf{0} \otimes \mathbf{0} = \mathbf{0}$, $\mathbf{0} \otimes \mathbf{1} = \mathbf{1}$ and $\mathbf{1} \otimes \mathbf{1} = \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}$. Thus, in the decomposition of a generic tensor $T^{\mu\nu}$ in representations of the rotation group, the $j = 0$ representation appears twice, the $j = 1$ representation appears three times, and the $j = 2$ once.

It is interesting to see how these representations are shared between the symmetric traceless, the trace and the antisymmetric part of the tensor $T^{\mu\nu}$, since these are the irreducible Lorentz representations. The trace is a Lorentz scalar, so it is in particular scalar under rotations and therefore is a $\mathbf{0}$ representation. An antisymmetric tensor $A^{\mu\nu}$ has six components, which can be written as A^{0i} and $(1/2)\epsilon^{ijk} A^{jk}$. These are two spatial vectors and therefore

$$A^{\mu\nu} \in \mathbf{1} \oplus \mathbf{1} \,. \tag{2.42}$$

For example, an important antisymmetric tensor in electromagnetism is the field strength tensor $F_{\mu\nu}$, and in this case the two vectors are $E^i = -F^{0i}$ and $B^i = -(1/2)\epsilon^{ijk} F^{jk}$, i.e. the electric and magnetic fields. Another example of an antisymmetric tensor is given by the Lorentz generators $J^{\mu\nu}$ themselves; in this case the two spatial vectors are the angular momentum and the boost generators that have been introduced in eq. (2.26).

Since we have identified the trace S with a $\mathbf{0}$ and $A^{\mu\nu}$ with $\mathbf{1} \oplus \mathbf{1}$, comparison with eq. (2.41) shows that the nine components of a symmetric traceless tensor $S^{\mu\nu}$ decompose, from the point of view of spatial rotations, as

$$S^{\mu\nu} \in \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2} \,. \tag{2.43}$$

Observe that, when in eq. (2.41) we write $T^{\mu\nu}$ as $(\mathbf{0} \oplus \mathbf{1}) \otimes (\mathbf{0} \oplus \mathbf{1})$, the first $\mathbf{0}$ corresponds to taking the index $\mu = 0$, the first $\mathbf{1}$ corresponds to taking the index $\mu = i$, and similarly for the second factor $(\mathbf{0} \oplus \mathbf{1})$ and the index ν. Therefore the term $(\mathbf{0} \otimes \mathbf{0})$ in eq. (2.41) corresponds to T^{00}, $(\mathbf{0} \otimes \mathbf{1})$ is T^{0i}, $(\mathbf{1} \otimes \mathbf{0})$ is T^{i0} and $(\mathbf{1} \otimes \mathbf{1})$ is T^{ij}. It is clear that T^{00} is a scalar under spatial rotations, while T^{0i} and T^{i0} are spatial vectors. As for T^{ij}, the antisymmetric part $A^{ij} = T^{ij} - T^{ji}$ is a vector, as can be seen considering $\epsilon^{ijk}A^{jk}$; this gives the third **1** representation. The symmetric part $S^{ij} = T^{ij} + T^{ji}$ can be separated into its trace, which gives the second **0** representation, and the traceless symmetric part, which therefore must have $j = 2$. For example, gravitational waves can be described by a traceless symmetric spatial tensor (transverse to the propagation direction) and therefore have spin 2, see Exercise 2.6.

In general, a symmetric tensor with N indices contains angular momenta up to $j = N$. In four dimensions, higher antisymmetric tensors are instead less interesting, because the index μ takes only four values $0, \ldots, 3$ and therefore we cannot antisymmetrize over more than four indices, otherwise we get zero. Furthermore, a totally antisymmetric tensor with four indices, $A^{\mu\nu\rho\sigma}$, has only one independent component A^{0123}, so it must be a Lorentz scalar. An antisymmetric tensor with three indices, $A^{\mu\nu\rho}$, has $4 \cdot 3 \cdot 2/3! = 4$ components and it has the same transformation properties of a four-vector.

The last point can be better understood introducing the totally antisymmetric tensor defined as follows. In a given reference frame $\epsilon^{\mu\nu\rho\sigma}$ is defined by $\epsilon^{0123} = +1$ and by the condition of total antisymmetry, so it vanishes if any two indices are equal and it changes sign for any exchange of indices, e.g. $\epsilon^{1023} = -1$, etc. Normally, if one gives the numerical value of the components of a tensor in a given frame, in another frame they will be different. The ϵ tensor is however special, because under (proper) Lorentz transformations

$$\epsilon^{\mu\nu\rho\sigma} \to \Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'}\Lambda^{\rho}{}_{\rho'}\Lambda^{\sigma}{}_{\sigma'}\epsilon^{\mu'\nu'\rho'\sigma'} = (\det\Lambda)\epsilon^{\mu\nu\rho\sigma} = \epsilon^{\mu\nu\rho\sigma} . \quad (2.44)$$

So, the components of the ϵ tensor have the same numerical value in all Lorentz frames. In terms of this tensor, it is immediate to understand that the four independent components of $A^{\mu\nu\rho}$ can be rearranged in a four-vector $A_\mu = \epsilon_{\mu\nu\rho\sigma}A^{\nu\rho\sigma}$, and that $A^{0123} = (1/4!)\epsilon_{\mu\nu\rho\sigma}A^{\mu\nu\rho\sigma}$ is a scalar.

A tensor which is invariant under all group transformations (i.e. for the Lorentz group, a tensor which has the same form in all Lorentz frames) is called an *invariant tensor*. The only other invariant tensor of the Lorentz group is $\eta_{\mu\nu}$; its invariance follows from the defining property of the Lorentz group, eq. (2.13).

2.5 Spinorial representations

2.5.1 Spinors in non-relativistic quantum mechanics

Tensor representations do not exhaust all physically interesting finite-dimensional representations of the Lorentz group. We can understand the issue considering spatial rotations, i.e. the $SO(3)$ subgroup of the Lorentz group. The tensor representations of $SO(3)$ are constructed exactly as before, with scalars ϕ, spatial vectors v^i, tensors T^{ij}, etc. with

$i = 1, 2, 3$. However we know from non-relativistic quantum mechanics that, beside the tensor representations, there are other representations of great physical interest. These are the spinorial representations. Strictly speaking, these are not $SO(3)$ representations, because under a rotation of 2π a spinor changes sign, while an $SO(3)$ rotation by 2π is the same as the identity transformation. However, since the observables are quadratic in the wave function, this sign ambiguity is perfectly acceptable physically, and these representations must be included. In more formal terms, this means that, for spatial rotations, the physically relevant group is not $SO(3)$ but rather $SU(2)$.

We recall some facts about $SU(2)$ representations, well known from non-relativistic quantum mechanics. The Lie algebras of $SU(2)$ and of $SO(3)$ are the same, and are given by the angular momentum algebra

$$[J^i, J^j] = i\epsilon^{ijk} J^k . \tag{2.45}$$

From the discussion in Section 2.1, we see that the Lie algebra knows only about the properties of a group near the identity element, and the fact that $SU(2)$ and $SO(3)$ have the same Lie algebra means that they are indistinguishable at the level of infinitesimal transformations. However, $SU(2)$ and $SO(3)$ differ at the global level, i.e. far from the identity. In $SO(3)$ a rotation by 2π is the same as the identity. Instead, it can be shown that $SU(2)$ is periodic only under rotations by 4π. This means that an object that picks a minus sign under a rotation by 2π is an acceptable representation of $SU(2)$, while it is not an acceptable representation of $SO(3)$. Therefore when we consider $SU(2)$ we include the solutions of eq. (2.45) that correspond to half-integer spin, while for $SO(3)$ we only retain representations with integer spin. Thus, the representations of $SU(2)$ are labeled by an index j which takes values $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and gives the spin of the state, in units of $\hbar$. The spin-j representation has dimension $2j + 1$, and the various states within it are labeled by j_z, which takes the values $-j, \ldots, j$ in integer steps. The representation $j = 1/2$ is called the spinorial representation, and has dimension 2: on it the J^i are represented as

$$J^i = \frac{\sigma^i}{2} , \tag{2.46}$$

where σ^i are the Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{2.47}$$

They satisfy the algebraic identity

$$\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k , \tag{2.48}$$

from which it follows immediately that $\sigma^i/2$ obey the commutation relations (2.45).

The spinorial is the fundamental representation of $SU(2)$ since all representations can be constructed with tensor products of spinors. In physical terms, this means that with spin 1/2 particles we can construct composite systems with all possible integer or half-integer spin. For instance, the composition of two spin 1/2 states gives spin zero and spin 1,

$$\frac{\mathbf{1}}{\mathbf{2}} \otimes \frac{\mathbf{1}}{\mathbf{2}} = \mathbf{0} \oplus \mathbf{1}\,. \tag{2.49}$$

If we denote by $\uparrow$ and $\downarrow$ the $j = 1/2$ states with $j_z = +1/2$ and $j_z = -1/2$, respectively, then the three states with $j = 1$ are given by

$$(\uparrow\uparrow)\,, \quad \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)\,, \quad (\downarrow\downarrow) \tag{2.50}$$

while the singlet (i.e. the scalar state) is

$$\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\,. \tag{2.51}$$

2.5.2 Spinors in the relativistic theory

We certainly want to keep spinors in the relativistic theory. This means that we must enlarge the set of representations of the Lorentz group, compared to the tensor representations discussed above. This is most easily done starting from the Lorentz algebra in the form given by eqs. (2.27)–(2.29), and defining

$$\mathbf{J}^{\pm} = \frac{\mathbf{J} \pm i\mathbf{K}}{2}\,. \tag{2.52}$$

The Lie algebra becomes

$$\left[J^{+,i}, J^{+,j}\right] = i\epsilon^{ijk} J^{+,k} \tag{2.53}$$

$$\left[J^{-,i}, J^{-,j}\right] = i\epsilon^{ijk} J^{-,k} \tag{2.54}$$

$$\left[J^{+,i}, J^{-,j}\right] = 0\,. \tag{2.55}$$

Therefore we have two copies of the angular momentum algebra, which commute between themselves.[7]

Having written the Lorentz group in this form, it is now easy to include spinorial representations: we simply take all solutions of the algebra (2.53)–(2.55), including spinor representations.

Since we know the representations of $SU(2)$, and here we have two commuting $SU(2)$ factors, we find that:

- The representations of the Lorentz algebra can be labeled by two half-integers: (j_-, j_+).
- The dimension of the representation (j_-, j_+) is $(2j_- + 1)(2j_+ + 1)$.
- The generator of rotations $\mathbf{J}$ is related to $\mathbf{J}^+$ and $\mathbf{J}^-$ by $\mathbf{J} = \mathbf{J}^+ + \mathbf{J}^-$; therefore, by the usual addition of angular momenta in quantum mechanics, in the representation (j_-, j_+) we have states with all possible spin j in integer steps between the values $|j_+ - j_-|$ and $j_+ + j_-$.

[7] The fact that the Lorentz algebra can be written as the algebra of $SU(2) \times SU(2)$ does *not* mean that the Lorentz group $SO(3,1)$ is the same as $SU(2) \times SU(2)$. First of all, the Lie algebra only reflects the properties of the group close to the identity. Furthermore, $\mathbf{J}^{\pm}$ are complex combinations of $\mathbf{J}$ and $\mathbf{K}$. Observe that, because of the factor i in eq. (2.52), a representation of $SU(2) \times SU(2)$ with $\mathbf{J}^{\pm}$ hermitian induces a representation of $SO(3,1)$ with $\mathbf{J}$ hermitian but $\mathbf{K}$ antihermitian. For the more mathematical reader: $SU(2) \times SU(2)$ is the universal covering group of $SO(4)$ (similarly to the fact that $SU(2)$ is the universal covering group of $SO(3)$) and $SO(4)$ is the Euclidean version of the Lorentz group, i.e. it is obtained taking the time variable t purely imaginary. The universal covering group of $SO(3,1)$ is $SL(2,C)$.

The representations are in general complex and the dimension of the representation is the number of independent complex components. In some cases we can impose a reality condition and $(2j_- + 1)(2j_+ + 1)$ becomes the number of independent real components. The representations (j_-, j_+) must include all tensor representations discussed in the previous section, plus spinorial representations. We examine the simplest cases.

$(\mathbf{0}, \mathbf{0})$. This representation has dimension one. On it, $\mathbf{J}^{\pm} = 0$ so also $\mathbf{J}, \mathbf{K}$ are zero. Therefore it is the scalar representation.

$(\frac{1}{2}, \mathbf{0})$ and $(\mathbf{0}, \frac{1}{2})$. These representations have both dimension two and spin 1/2, so they are spinorial representations. We denote by $(\psi_L)_\alpha$, with $\alpha = 1, 2$, a spinor in $(1/2, 0)$ and by $(\psi_R)_\alpha$ a spinor in $(0, 1/2)$ (sometimes in the literature the index of ψ_L is instead denoted by $\dot{\alpha}$ to stress that it is an index in a different representation compared to the index of ψ_R). ψ_L is called a *left-handed Weyl spinor* and ψ_R is called a *right-handed Weyl spinor*:

$$\text{Weyl spinors:} \quad \psi_L \in \left(\frac{1}{2}, 0\right), \quad \psi_R \in \left(0, \frac{1}{2}\right). \tag{2.56}$$

We want to determine the explicit form of the generators $\mathbf{J}, \mathbf{K}$ on Weyl spinors. Consider first the representation $(1/2, 0)$. By definition, on this representation $\mathbf{J}^-$ is represented by a 2×2 matrix, while $\mathbf{J}^+ = 0$. The solution of (2.54) in terms of 2×2 matrices is of course $\mathbf{J}^- = \boldsymbol{\sigma}/2$, and therefore

$$\mathbf{J} = \mathbf{J}^+ + \mathbf{J}^- = \frac{\boldsymbol{\sigma}}{2} \tag{2.57}$$

$$\mathbf{K} = -i(\mathbf{J}^+ - \mathbf{J}^-) = i\,\frac{\boldsymbol{\sigma}}{2}\,. \tag{2.58}$$

Observe that in this representation the generators K^i are not hermitian, in agreement with the comment in note 7. This is a consequence of the fact that the Lorentz group is non-compact and of the theorem mentioned on page 16, which states that non-compact groups have no unitary representations of finite dimension, except for representations in which the non-compact generators (in this case the K^i) are represented trivially, i.e. $K^i = 0$. We can now write explicitly how a Weyl spinor transforms under Lorentz transformations, using eq. (2.31),

$$\psi_L \to \Lambda_L \psi_L = \exp\left\{(-i\boldsymbol{\theta} - \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\right\} \psi_L\,. \tag{2.59}$$

Repeating the argument for the $(0, 1/2)$ representation, we find again $\mathbf{J} = \boldsymbol{\sigma}/2$ but $\mathbf{K} = -i\boldsymbol{\sigma}/2$ and

$$\psi_R \to \Lambda_R \psi_R = \exp\left\{(-i\boldsymbol{\theta} + \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\right\} \psi_R\,. \tag{2.60}$$

Note that $\Lambda_{L,R}$ are complex matrices, and therefore necessarily the two components of a Weyl spinor are complex numbers.

Using the property of the Pauli matrices $\sigma^2\sigma^i\sigma^2 = -\sigma^{i*}$ and the explicit form of $\Lambda_{L,R}$ it is easy to show that

$$\sigma^2\Lambda_L^*\sigma^2 = \Lambda_R\,. \tag{2.61}$$

From this it follows that

$$\sigma^2\psi_L^* \to \sigma^2(\Lambda_L\psi_L)^* = (\sigma^2\Lambda_L^*\sigma^2)\sigma^2\psi_L^* = \Lambda_R(\sigma^2\psi_L^*) \tag{2.62}$$

where we used the fact that $\sigma^2\sigma^2 = 1$. Therefore , if $\psi_L \in (1/2, 0)$, then $\sigma^2\psi_L^*$ is a right-handed Weyl spinor,

$$\boxed{\sigma^2\psi_L^* \in \left(0, \frac{1}{2}\right)\,.} \tag{2.63}$$

We define the operation of charge conjugation on Weyl spinors as an operation that transforms ψ_L into a new spinor ψ_L^c defined as

$$\psi_L^c = i\sigma^2\psi_L^*\,. \tag{2.64}$$

Then charge conjugation transforms a left-handed Weyl spinor into a right-handed one. Taking the complex conjugate of eq. (2.64) and denoting the right-handed spinor ψ_L^c by ψ_R, we have $\psi_L = -i\sigma^2\psi_R^*$ (having used the fact that σ^2 is purely imaginary and $\sigma^2\sigma^2 = 1$). Therefore we define charge conjugation on a right-handed spinor ψ_R as

$$\psi_R^c = -i\sigma^2\psi_R^*\,, \tag{2.65}$$

so that charge conjugation transforms a right-handed Weyl spinor into a left-handed one. The factor i in eq. (2.64) is chosen so that, iterating the transformation twice, we get the identity operation,

$$(\psi_L^c)^c = (i\sigma^2\psi_L^*)^c = -i\sigma^2(i\sigma^2\psi_L^*)^* = \psi_L\,. \tag{2.66}$$

We will understand the physical meaning of charge conjugation in Chapter 4.

$(\frac{1}{2}, \frac{1}{2})$. This representation has (complex) dimension four and $|1/2 - 1/2| \leqslant j \leqslant 1/2 + 1/2$, i.e. $j = 0, 1$. Comparing with eq. (2.40) we see that it is a complex four-vector representation. A generic element of the $(1/2, 1/2)$ representation can be written as a pair $((\psi_L)_\alpha, (\xi_R)_\beta)$, where ψ_L and ξ_R are two independent Weyl spinors, left-handed and right-handed, respectively, and α, β take the values $1, 2$. We want to make explicit the relation between these four (complex) quantities and the four components of a (complex) four-vector.

First of all, we have seen above that, given a right-handed spinor ξ_R, we can form a left-handed spinor $\xi_L \equiv -i\sigma^2\xi_R^*$, and similarly from ψ_L we can build $\psi_R \equiv i\sigma^2\psi_L^*$. We define the matrices σ^μ and $\bar{\sigma}^\mu$ as

$$\sigma^\mu = (1, \sigma^i)\,, \qquad \bar{\sigma}^\mu = (1, -\sigma^i)\,, \tag{2.67}$$

where σ^i are the Pauli matrices and 1 is the 2×2 identity matrix. Then, it is easy to show (see Exercise 2.3) that

$$\xi_R^\dagger \sigma^\mu \psi_R \tag{2.68}$$

and

$$\xi_L^\dagger \bar{\sigma}^\mu \psi_L \,. \tag{2.69}$$

are contravariant four-vectors. These four vectors are by construction complex. Since the matrix $\Lambda^\mu{}_\nu$ that represents the Lorentz transformation of a four-vector is real, given a complex four-vector V^μ it is consistent with Lorentz invariance to impose on it a reality condition, $V_\mu = V_\mu^*$ because, if we impose it in a given frame, it will remain true in all Lorentz frames. Therefore we obtain the real four-vector representation.

(1, 0) and **(0, 1)**. These correspond to self-dual and anti-self-dual antisymmetric tensors $A^{\mu\nu}$, and each have complex dimension three, i.e. real dimension six. We discuss them in Exercise 2.5.

2.6 Field representations

Our main motivation for studying Lorentz symmetry is to construct a Lorentz-invariant field theory. A field $\phi(x)$ is a function of the coordinates with some definite transformation properties under the Lorentz group. In general, if

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{2.70}$$

the field $\phi(x)$ will transform into a new function of the new coordinates,

$$\phi(x) \to \phi'(x') \,. \tag{2.71}$$

To define how a field transforms means to state how $\phi'(x')$ is related to $\phi(x)$.

2.6.1 Scalar fields

The simplest possible transformation is that of a *scalar field*,

$$\boxed{\phi'(x') = \phi(x) \,.} \tag{2.72}$$

In other words, the numerical value of a scalar field at a point is Lorentz invariant: a point P has coordinates x in a reference frame and x' in the transformed frame, and the functional form of the field changes so that its numerical value in P is the same, independently of how P is labeled.

Consider now an infinitesimal Lorentz transformation

$$x^\rho \to x'^\rho = x^\rho + \delta x^\rho \tag{2.73}$$

with

$$\delta x^\rho = \omega^\rho{}_\sigma x^\sigma = -\frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^\rho{}_\sigma\, x^\sigma\,, \tag{2.74}$$

and $(J^{\mu\nu})^\rho{}_\sigma = i(\eta^{\mu\rho}\,\delta^\nu_\sigma - \eta^{\nu\rho}\,\delta^\mu_\sigma)$, as in eqs. (2.23) and (2.24). Under this transformation $\delta\phi \equiv \phi'(x') - \phi(x) = 0$ by definition. This corresponds to the fact that the scalar representation gives a trivial representation of the generators, $J^{\mu\nu} = 0$. However, in the case of fields we have a more interesting possibility, namely we can consider an infinitesimal variation *at fixed coordinate* x (rather than at a given point P),

$$\delta_0\phi \equiv \phi'(x) - \phi(x)\,. \tag{2.75}$$

To understand the difference between $\delta\phi$ and $\delta_0\phi$ we observe that, when we compute $\delta\phi = \phi'(x') - \phi(x)$, we are studying how a single degree of freedom (the field evaluated at the point P) changes when we change the label of the point P from x to x'. However the point P is kept fixed, so the base space is made by the single degree of freedom $\phi(P)$ and therefore is one-dimensional. More generally, when in the next subsections we consider spinor or vector fields, we will see that $\delta\psi$ or δA_μ always provides a finite-dimensional representation of the generators. For instance the four degrees of freedom $A_\mu(P)$ provide a four-dimensional base space. Instead, when we compute $\delta_0\phi$, we are comparing the fields at two different space-time points P and P', so we are comparing different degrees of freedom. The base space now becomes the set of $\phi(P)$ with P varying over all of space-time, or in other words is a space of functions, and therefore it is an infinite-dimensional base-space. We then obtain an infinite-dimensional representation of the generators.

To find the generators in this representation, we expand eq. (2.75) to first order in δx,

$$\delta_0\phi = \phi'(x' - \delta x) - \phi(x) = -\delta x^\rho \partial_\rho \phi(x)\,. \tag{2.76}$$

Using eq. (2.74) for δx^ρ, this can be rewritten as

$$\delta_0\phi = \frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^\rho{}_\sigma\, x^\sigma \partial_\rho\phi \equiv -\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\phi\,, \tag{2.77}$$

where we defined

$$L^{\mu\nu} = -(J^{\mu\nu})^\rho{}_\sigma\, x^\sigma\partial_\rho = i(x^\mu\partial^\nu - x^\nu\partial^\mu)\,. \tag{2.78}$$

We can easily check that the operators $L^{\mu\nu}$ satisfy the Lie algebra (2.25) and therefore give a representation of the generators of the Lorentz group. As discussed above, the basis for the representation is the space of scalar fields. This is a space of functions, so it is infinite-dimensional, and therefore this is an *infinite-dimensional representation* of the Lorentz algebra. We have not yet specified what is the scalar product in the field space, so we cannot yet ask whether this representation is unitary. We postpone the issue to the next chapter.

Recalling that with our metric signature $p^\mu = +i\partial^\mu$ (see the Notation), we find $L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$. In particular, for spatial rotations we have $L^{ij} = x^i p^j - x^j p^i$ and $L^i = (1/2)\epsilon^{ijk}L^{jk} = \epsilon^{ijk}x^j p^k$, and we recognize that L^i is the orbital angular momentum.

2.6.2 Weyl fields

A left-handed Weyl field $\psi_L(x)$ is defined as a field that, under $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$, transforms as

$$\boxed{\psi_L(x) \to \psi'_L(x') = \Lambda_L \psi_L(x)\,,} \tag{2.79}$$

with Λ_L given by eq. (2.59). Similarly a right-handed Weyl field ψ_R transforms with Λ_R given in eq. (2.60). In the classical theory we will consider ψ_L, ψ_R as ordinary, commuting, c-numbers.

The representation of the Lorentz generators on ψ_L can be found computing

$$\begin{aligned}\delta_0\psi_L &\equiv \psi'_L(x) - \psi_L(x) = \psi'_L(x' - \delta x) - \psi_L(x)\\ &= \psi'_L(x') - \delta x^\rho \partial_\rho \psi_L(x) - \psi_L(x)\\ &= (\Lambda_L - 1)\psi_L(x) - \delta x^\rho \partial_\rho \psi_L(x)\,.\end{aligned} \tag{2.80}$$

We see that $\delta_0\psi_L$ is made of two parts; one comes from the variation of the coordinate δx^ρ and is the same as for scalar fields. Exactly as in eqs. (2.76) and (2.77), we have

$$-\delta x^\rho \partial_\rho \psi_L = -\frac{i}{2}\omega_{\mu\nu} L^{\mu\nu} \psi_L\,, \tag{2.81}$$

with $L^{\mu\nu}$ given in eq. (2.78). We write Λ_L in the form

$$\Lambda_L = e^{-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}}\,. \tag{2.82}$$

Then eq. (2.80) becomes

$$\delta_0\psi_L = -\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu} \psi_L \tag{2.83}$$

with

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}\,. \tag{2.84}$$

Comparing eq. (2.82) with eq. (2.59) we see that

$$S^i = \frac{1}{2}\epsilon^{ijk} S^{jk} = \frac{\sigma^i}{2}\,, \tag{2.85}$$

while

$$S^{i0} = i\frac{\sigma^i}{2}\,. \tag{2.86}$$

We recognize in eq. (2.84) the separation of the angular momentum into the orbital and the spin contributions. It is clear that this separation is completely general, and holds for any representation. The orbital part $L^{\mu\nu}$ always has the form (2.78) independently of the representation, while $S^{\mu\nu}$ depends on the specific representation used. For instance, for right-handed Weyl fields S^i are still given by eq. (2.85) while $S^{i0} = -i\sigma^i/2$, as we see from eq. (2.60).

2.6.3 Dirac fields

Consider a parity transformation $(t, \mathbf{x}) \to (t, -\mathbf{x})$. Under this operation the boost generators behave as true vectors and change sign, $\mathbf{K} \to -\mathbf{K}$, since the parity transformation reverses the velocity $\mathbf{v}$ of the boost. The angular momentum generator is instead a pseudovector, $\mathbf{J} \to \mathbf{J}$. Therefore a parity operation exchanges the $J^i_\pm$ generators, $J^i_+ \leftrightarrow J^i_-$. This means that under a parity transformation an object in the (j_-, j_+) representation is transformed into an object in the (j_+, j_-) representation. Therefore the representation (j_-, j_+) of the Lorentz group is not at the same time a basis for a representation of the parity transformation, unless $j_- = j_+$. In particular, ψ_L and ψ_R, separately, are *not* a basis for a representation of the parity transformation.

In Nature, we know experimentally that parity is violated by weak interactions. At the theoretical level, this is reflected in the fact that in the Standard Model the left and right-handed components of the spin 1/2 particles enter the theory in a very different way, as we will see in Chapter 8. However, we saw in Section 1.2 that the typical scale of weak interactions is $O(100)$ GeV, much higher than the scale of strong and of electromagnetic interactions. At sufficiently low energies, therefore, the effect of weak interactions is small, and the dominant contributions come from the electromagnetic and the strong interactions, which both conserve parity. In this case, it is convenient to work with fields which provide a representation of Lorentz *and* parity transformations. We then define a *Dirac field* as[8]

$$\boxed{\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}.} \tag{2.87}$$

[8] More precisely, this is the Dirac field written in the chiral basis, see Section 3.4.2.

A Dirac field therefore has four complex components, and it provides a basis for a representation of both Lorentz and parity transformations. In fact, under a Lorentz transformation, $\Psi \to \Lambda_D \Psi$ with

$$\Lambda_D = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix}, \tag{2.88}$$

and Λ_L, Λ_R given in eqs. (2.59) and (2.60).[9] Under a parity transformation P the coordinates change as $x^\mu \to x'^\mu = (t, -\mathbf{x})$ while

$$\begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} \to \begin{pmatrix} \psi_R(x') \\ \psi_L(x') \end{pmatrix} \tag{2.89}$$

and therefore

$$\Psi(x) \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi(x'). \tag{2.90}$$

[9] In Section 3.4.2, after introducing the Dirac matrices, we will see how to write Λ_D in terms of the commutator of Dirac matrices, and the result will be independent of the chiral basis that we have used here.

When we study the *quantized* Dirac field we will examine the possibility and the meaning of an overall phase $\eta = \pm 1$ in the transformation law (2.90), see Section 4.2.3.

In eqs. (2.64) and (2.65) we defined the operation of charge conjugation on Weyl spinors. Given a Dirac spinor Ψ as in eq. (2.87), charge conjugation allows us to define a new Dirac spinor

$$\Psi^c = \begin{pmatrix} -i\sigma^2\psi_R^* \\ i\sigma^2\psi_L^* \end{pmatrix} = -i\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\Psi^* . \tag{2.91}$$

and, as for Weyl spinors, iterating charge conjugation twice one finds the identity transformation,

$$(\Psi^c)^c = \Psi . \tag{2.92}$$

Note that the coordinates x^μ are unchanged under charge conjugation. We will understand the importance of charge conjugation when we quantize the theory and we will find particles and antiparticles.

Dirac spinors are the basic objects in quantum electrodynamics (QED). Since QED preserves parity and charge conjugation, the Weyl spinors always appear in the combination Ψ. On Ψ parity is a well-defined operation, and we can use it to construct a parity-invariant theory while, having for instance only ψ_L at our disposal, it is impossible to build a theory invariant under parity. We will see that in the Standard Model, parity and charge conjugation are not symmetries and ψ_L, ψ_R appear separately, in a non-symmetric way. Therefore, Weyl spinors are more fundamental objects than Dirac spinors.

2.6.4 Majorana fields

A Majorana spinor is a Dirac spinor in which ψ_L and ψ_R are not independent, but rather $\psi_R = i\sigma^2\psi_L^*$,

$$\Psi_M = \begin{pmatrix} \psi_L \\ i\sigma^2\psi_L^* \end{pmatrix} . \tag{2.93}$$

So, it has the same number of degrees of freedom as a Weyl spinor, although it is written in the form of a Dirac spinor. From this definition it follows that a Majorana spinor is invariant under charge conjugation

$$\Psi_M^c = \Psi_M . \tag{2.94}$$

Observe that, if we have a complex scalar field $\phi(x)$, we can impose on it a reality condition $\phi(x) = \phi^*(x)$, and this is a Lorentz-invariant condition: since ϕ and ϕ^* are both Lorentz invariant, if we impose $\phi = \phi^*$ in a frame, we will have $\phi = \phi^*$ in any other frame. The same is true for the four-vector representation, as we already discussed on page 29. For a Dirac spinor Ψ the situation is different; Ψ is a complex field, and the condition $\Psi = \Psi^*$ is not Lorentz invariant, since the matrix Λ_D in eq. (2.88) is not real. Therefore, if we impose the relation $\Psi = \Psi^*$ in a frame, it will not hold in general in another Lorentz frame. Instead, the condition (2.94) is by construction Lorentz invariant, since it is a consequence of the definition (2.93), which in turns expresses the

Lorentz-invariant statement that $i\sigma^2\psi_L^*$ is a right-handed spinor. Since Ψ_M^c involves complex conjugation, see eq. (2.91), the condition (2.94) is a Lorentz-invariant relation between Ψ and Ψ^*, and in this sense it is called a reality condition.

So we can see Majorana fields as "real" Dirac fields, with respect to the only possible Lorentz-invariant reality condition, eq. (2.94).

It is possible that Majorana spinors play an important role in the description of the neutrino. We will come to this issue later.

2.6.5 Vector fields

The definition of vector fields at this point is obvious. A (contravariant) vector field $V^\mu(x)$ is defined as a field that, under $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$, transforms as

$$V^\mu(x) \to V'^\mu(x') = \Lambda^\mu{}_\nu V^\nu(x) \,. \tag{2.95}$$

From the discussion in Section 2.4.1 we see that a general vector field has a spin-0 and a spin-1 component. An example of a vector field that will be important for us is the gauge field $A^\mu(x)$ in electromagnetism. We will see in Section 4.3.1 that $A^\mu(x)$ is subject to some conditions, stemming from gauge invariance, that eliminate the spin-0 component and the state with $(j = 1, j_z = 0)$, where z is the propagation direction.

Since a vector field belongs to the $(1/2, 1/2)$ representation, it has $j_- = j_+$ and therefore it is a basis for the representation of parity. A true vector transform as $(V^0, \mathbf{V}) \to (V^0, -\mathbf{V})$ while a pseudovector (or axial vector) transforms as $(V^0, \mathbf{V}) \to (-V^0, \mathbf{V})$.

Tensor fields are defined similarly.

2.7 The Poincaré group

Beside invariance under Lorentz transformations, we require also invariance under space-time translations. A generic element of the translation group is written as

$$\exp\{-iP^\mu a_\mu\} \tag{2.96}$$

where a_μ are the parameters of the translation, $x^\mu \to x^\mu + a^\mu$, and the components of the four-momentum operator P^μ are the generators. Translations plus Lorentz transformations form a group, called the *Poincaré group*, or the inhomogeneous Lorentz group (it is sometimes denoted as $ISO(3,1)$, where "I" stands for inhomogeneous).

Since the translations commute, we have

$$[P^\mu, P^\nu] = 0 \,. \tag{2.97}$$

To find the commutator between P^μ and $J^{\rho\sigma}$ we can start from the commutators

$$[J^i, P^j] = i\epsilon^{ijk} P^k \,, \tag{2.98}$$

$$[J^i, P^0] = 0 \,, \tag{2.99}$$

which express the facts that P^i is a vector under rotations and that the energy is a scalar under rotations. The unique Lorentz-covariant generalization of eqs. (2.98) and (2.99) is

$$[P^\mu, J^{\rho\sigma}] = i\,(\eta^{\mu\rho}P^\sigma - \eta^{\mu\sigma}P^\rho)\,. \qquad (2.100)$$

Together with the Lorentz algebra (2.25), eqs. (2.97) and (2.100) define the Poincaré algebra. In terms of $J^i, K^i, P^0 = H$ and P^i it reads

$$[J^i, J^j] = i\epsilon^{ijk}J^k\,, \quad [J^i, K^j] = i\epsilon^{ijk}K^k\,, \quad [J^i, P^j] = i\epsilon^{ijk}P^k\,, \qquad (2.101)$$

$$[K^i, K^j] = -i\epsilon^{ijk}J^k\,, \quad [P^i, P^j] = 0\,, \quad [K^i, P^j] = iH\delta^{ij}\,, \qquad (2.102)$$

$$[J^i, H] = 0\,, \quad [P^i, H] = 0\,, \quad [K^i, H] = iP^i\,. \qquad (2.103)$$

Equations (2.101) express the fact that the J^i generate spatial rotations and K^i, P^i are vectors under rotations. Equations (2.103) state that J^i and P^i commute with the generator of time translations and therefore are conserved quantities; the K^i instead are not conserved, and this is the reason why the eigenvalues of $\mathbf{K}$ are not used for labeling physical states.

2.7.1 Representation on fields

We saw in Section 2.6 that fields provide an infinite-dimensional representation of the Lorentz group, and that on fields the generators $J^{\mu\nu}$ are represented as

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu} \qquad (2.104)$$

where

$$L^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu) \qquad (2.105)$$

and $S^{\mu\nu}$ depends on the spin of the field in question, but not on the coordinates x^μ. To obtain a representation of the full Poincaré group on fields, we must now find how to represent the four-momentum operator P^μ, i.e. we have to specify the transformation law of fields under translations.

We require that all fields, independently of their transformation property under the Lorentz group, behave as *scalars* under space-time translation. Let us label by ϕ a generic field, either a Lorentz scalar field, or a component of a spinor field ξ_α with α given, or a given component V^μ of a vector field, etc. Then, under a translation $x \to x' = x + a$, all fields, independently of their Lorentz properties, transform as

$$\phi'(x') = \phi(x)\,. \qquad (2.106)$$

Under an infinitesimal translation $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu$ we have, to first order in ϵ,

$$\delta_0\phi \equiv \phi'(x) - \phi(x) = \phi'(x' - \epsilon) - \phi(x) = -\epsilon^\mu\partial_\mu\phi(x)\,. \qquad (2.107)$$

On the other hand, from the form (2.96) of the translation operator, it follows that

$$\phi'(x'-\epsilon) = e^{-iP^\mu(-\epsilon_\mu)}\phi'(x') = e^{i\epsilon_\mu P^\mu}\phi(x) \tag{2.108}$$

and therefore to first order in ϵ

$$\delta_0\phi = i\epsilon_\mu P^\mu \phi(x)\,. \tag{2.109}$$

Comparing eqs. (2.107) and (2.109) we see that the momentum operator is represented as

$$P^\mu = +i\partial^\mu\,. \tag{2.110}$$

Therefore

$$H = i\frac{\partial}{\partial x^0} = i\frac{\partial}{\partial t}\,, \qquad P^i = i\partial^i = -i\partial_i = -i\frac{\partial}{\partial x^i}\,. \tag{2.111}$$

The explicit form of $J^{\mu\nu}$ and of P^μ has been found requiring that the fields have well-defined transformation properties under the Poincaré group; therefore these explicit expressions must automatically satisfy the Poincaré algebra. We can check this easily observing that $S^{\mu\nu}$ does not depend on the coordinates and therefore commutes with ∂^μ, while $[\partial^\mu, x^\nu] = \eta^{\mu\nu}$. Therefore

$$\begin{aligned}[P^\mu, J^{\rho\sigma}] &= [i\partial^\mu, i(x^\rho\partial^\sigma - x^\sigma\partial^\rho)] = -\eta^{\mu\rho}\partial^\sigma + \eta^{\mu\sigma}\partial^\rho \\ &= i\left(\eta^{\mu\rho}P^\sigma - \eta^{\mu\sigma}P^\rho\right)\,,\end{aligned} \tag{2.112}$$

in agreement with eq. (2.100). The commutator $[P^\mu, P^\nu] = 0$ is also satisfied by $P^\mu = i\partial^\mu$ and we already know that the commutator $[J^{\mu\nu}, J^{\rho\sigma}]$ is correctly reproduced, so the full Poincaré algebra is obeyed.

2.7.2 Representation on one-particle states

The representation of the Poincaré group on fields allows us to construct Poincaré invariant Lagrangians, as we will study in the next chapter. At the classical level, a Lagrangian description is all that we need in order to specify the dynamics of the system. At the quantum level, however, one of our aims will be to understand how the concept of particle emerges from field quantization. It is therefore useful to see how the Poincaré group can be represented using as a basis the Hilbert space of a free particle. We will denote the states of a free particle with momentum $\mathbf{p}$ as $|\mathbf{p}\,, s\rangle$, where s labels collectively all other quantum numbers. Since $\mathbf{p}$ is a continuous and unbounded variable, this base space is infinite-dimensional. A theorem by Wigner (see Weinberg (1995), Chapter 2) states that on this Hilbert space any symmetry transformation can be represented by a unitary operator.[10] Therefore in this base space a Poincaré transformation is represented by a unitary matrix, and the generators J^i, K^i, P^i and H by hermitian operators.

The representations are labeled by the Casimir operators. One is easily found, and is $P_\mu P^\mu$. On a one-particle state it has the value m^2,

[10] Actually there is also the possibility of an anti-unitary operator; the only symmetry transformation where this happens is time-reversal, and we postpone the definition of anti-unitary operators to Chapter 4.

where m is the mass of the particle. Using the commutation relations of the Poincaré group one can verify that there is a second Casimir operator given by $W_\mu W^\mu$, where

$$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma \tag{2.113}$$

is called the Pauli–Lubanski four-vector. To prove that $W_\mu W^\mu$ is a Casimir operator is straightforward. First of all, W^μ is clearly a four-vector, so $W_\mu W^\mu$ is Lorentz-invariant and therefore commutes with $J^{\mu\nu}$. From the explicit form it also follows that

$$[W^\mu, P^\nu] = 0\,, \tag{2.114}$$

(using eq. (2.100) and the antisymmetry of $\epsilon^{\mu\nu\rho\sigma}$), and then $W_\mu W^\mu$ commutes also with P^ν.

Since $W_\mu W^\mu$ is Lorentz-invariant, we can compute it in the frame that we prefer. If $m \neq 0$, it is convenient to choose the rest frame of the particle; in this frame $W^\mu = (-m/2)\epsilon^{\mu\nu\rho 0} J_{\nu\rho} = (m/2)\epsilon^{0\mu\nu\rho} J_{\nu\rho}$, so $W^0 = 0$ while

$$W^i = \frac{m}{2}\epsilon^{0ijk} J^{jk} = \frac{m}{2}\epsilon^{ijk} J^{jk} = mJ^i\,. \tag{2.115}$$

Therefore on a one-particle state with mass m and spin j we have

$$-W_\mu W^\mu = m^2 j(j+1)\,, \qquad (m \neq 0)\,. \tag{2.116}$$

If instead $m = 0$ the rest frame does not exist, but we can choose a frame where $P^\mu = (\omega, 0, 0, \omega)$; in this frame a straightforward computation gives $W^0 = W^3 = \omega J^3$, $W^1 = \omega(J^1 - K^2)$ and $W^2 = \omega(J^2 + K^1)$. Therefore

$$-W_\mu W^\mu = \omega^2[(K^2 - J^1)^2 + (K^1 + J^2)^2]\,, \qquad (m = 0)\,. \tag{2.117}$$

Comparing eqs. (2.116) and (2.117) we see that the limit $m \to 0$ is quite subtle, and we must study separately the massive and massless representations.

Massive representations: In this case on the one-particle states we have $P^\mu P_\mu = m^2$ while $W_\mu W^\mu = -m^2 j(j+1)$. We will restrict to m real[11] and positive. Therefore the representations are labeled by the mass m and by the spin j. We can understand this better observing that, if $m \neq 0$, with a Lorentz transformation we can bring P^μ into the form $P^\mu = (m, 0, 0, 0)$. This choice of P^μ still leaves us with the freedom of performing spatial rotations. In other words, the space of one-particle states with momentum $P^\mu = (m, 0, 0, 0)$ is still a basis for the representation of spatial rotations. The group of transformations which leaves invariant a given choice of P^μ is called the *little group*. In this case, since we want to include spinor representations, the little group is $SU(2)$. The massive representations are therefore labeled by the mass m and by the spin $j = 0, 1/2, 1, \ldots$, and states within each

[11] In principle there is also the possibility of representations with $m^2 < 0$, known as tachyons. In field theory the emergence of a tachyonic mode is the signal of an instability, and reflects the fact that we have expanded around the wrong vacuum, e.g. around a maximum rather than a minimum of a potential.

representation are labeled by $j_z = -j, -j+1, \ldots, j$. This means that *massive particles of spin j have $2j+1$ degrees of freedom.*

Massless representations: When $P^2 = 0$ the rest frame does not exist, but we can reduce P^μ to the form $P^\mu = (\omega, 0, 0, \omega)$. The little group is the set of Poincaré transformations that leaves this vector unchanged. One sees immediately that the rotations in the (x, y) plane leave this P^μ invariant; this is an $SO(2)$ group, generated by J^3.

This part is more technical and can be omitted at a first reading. Just assume that the little group is $SO(2)$ and skip the part written in smaller characters.

Furthermore there are two less evident Lorentz transformations that do not change P^μ; to find the most general solution, it is sufficient to restrict to infinitesimal Lorentz transformations $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu$, and to look for the most general matrix $\omega^{\mu\nu}$ which satisfies $\omega^{\mu\nu} = -\omega^{\nu\mu}$ (in order to have a Lorentz transformation) and

$$\omega^{\mu\nu} P_\nu = 0\,, \tag{2.118}$$

for $P_\nu = (\omega, 0, 0, -\omega)$. Therefore

$$\begin{pmatrix} 0 & \omega^{01} & \omega^{02} & \omega^{03} \\ -\omega^{01} & 0 & \omega^{12} & \omega^{13} \\ -\omega^{02} & -\omega^{12} & 0 & \omega^{23} \\ -\omega^{03} & -\omega^{13} & -\omega^{23} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = 0\,, \tag{2.119}$$

which gives $\omega^{03} = 0$, $\omega^{01} + \omega^{13} = 0$ and $\omega^{02} + \omega^{23} = 0$. Denoting $\omega^{01} = \alpha$, $\omega^{02} = \beta$ and $\omega^{12} = \theta$ we see that the most general Lorentz transformation that leaves P^μ invariant can be written as

$$\Lambda = e^{-i(\alpha A + \beta B + \theta C)} \tag{2.120}$$

where (lowering the second Lorentz index)

$$A^\mu{}_\nu = i \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad B^\mu{}_\nu = i \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{2.121}$$

and

$$C^\mu{}_\nu = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.122}$$

Comparison with eq. (2.23) shows that $C^\mu{}_\nu$ is nothing but $(J^3)^\mu{}_\nu$, i.e. the explicit expression of J^3 as a 4×4 matrix in the four-vector representation. Similarly we find that $A^\mu{}_\nu = (K^1 + J^2)^\mu{}_\nu$ and $B^\mu{}_\nu = (K^2 - J^1)^\mu{}_\nu$. These are just the combinations that appear in eq. (2.117), so in the massless case

$$-W_\mu W^\mu = \omega^2 (A^2 + B^2)\,. \tag{2.123}$$

Using the commutation rules of the Lorentz group, or directly the explicit expressions given above, one finds that the operators J^3, A and B close an algebra:

$$[J^3, A] = +iB\,, \qquad [J^3, B] = -iA\,, \qquad [A, B] = 0\,. \tag{2.124}$$

Formally this is the same algebra generated by the operators p^x, p^y and $L^z = xp^y - yp^x$, which describe the translations and rotations of a Euclidean plane, with A and B playing the role of the translation operators. This algebra is denoted by $ISO(2)$. The matrices $A^\mu{}_\nu$ and $B^\mu{}_\nu$ given in eq. (2.121) are not hermitian.[12] This is as it should be, since they are 4×4 matrices, and therefore are a finite-dimensional representation of non-compact Lorentz generators.

[12] They would be hermitian if we write them as $A^{\mu\nu}, B^{\mu\nu}$ and $C^{\mu\nu}$. However, δx^ρ is proportional to $\omega_{\mu\nu}(J^{\mu\nu})^\rho{}_\sigma x^\sigma$, so the representation is provided by the matrices with one upper and one lower index, and it is for these matrices that the algebra (2.124) holds.

We can however represent the algebra (2.124) taking as the base space the one-particle states with momentum $\mathbf{p}$. In this representation A and B are hermitian operators because of Wigner's theorem and, since they are commuting, they can be diagonalized simultaneously. We denote by a, b the respective eigenvalues. Then

$$A|\mathbf{p}\,; a, b\rangle = a|\mathbf{p}\,; a, b\rangle\,, \qquad B|\mathbf{p}\,; a, b\rangle = b|\mathbf{p}\,; a, b\rangle\,. \tag{2.125}$$

However, if a and b are non-zero, we can now find a continuous set of eigenvalues! Consider in fact the state

$$|\mathbf{p}\,; a, b, \theta\rangle \equiv e^{-i\theta J^3}|\mathbf{p}\,; a, b\rangle\,, \tag{2.126}$$

with θ an arbitrary angle. We have

$$A\, e^{-i\theta J^3}|\mathbf{p}\,; a, b\rangle = e^{-i\theta J^3}\left(e^{i\theta J^3} A e^{-i\theta J^3}\right)|\mathbf{p}\,; a, b\rangle\,. \tag{2.127}$$

Using the commutation rules (2.124) we find that

$$e^{i\theta J^3} A e^{-i\theta J^3} = A\cos\theta - B\sin\theta \tag{2.128}$$

(this can be proved expanding the exponentials in power series) and therefore

$$A|\mathbf{p}\,; a, b, \theta\rangle = (a\cos\theta - b\sin\theta)|\mathbf{p}\,; a, b, \theta\rangle\,, \tag{2.129}$$

and similarly

$$B|\mathbf{p}\,; a, b, \theta\rangle = (a\sin\theta + b\cos\theta)|\mathbf{p}\,; a, b, \theta\rangle\,. \tag{2.130}$$

This means that, unless $a = b = 0$, we find representations corresponding to massless particles with a continuous internal degree of freedom θ. These representations do not so far find physical applications, and we therefore restrict to states with $a = b = 0$. Since for massless particles we found $-W_\mu W^\mu = \omega^2(A^2 + B^2)$, on these states (and only on these states) we have $-W_\mu W^\mu = 0$, which agrees with the $m \to 0$ limit of eq. (2.116). On the states with $a = b = 0$ the little group is simply $SO(2)$ or, equivalently, $U(1)$.

As for any abelian group, the irreducible representations of $SO(2)$ are one-dimensional. The generator of the group $SO(2)$ of rotations in the (x, y) plane is the angular momentum J^3 and therefore the one-dimensional representations are labeled by the eigenvalue h of J^3; it represents the angular momentum in the direction of propagation of the particle (in this case, the z axis), and is called the *helicity*.

It can be shown that h is quantized, $h = 0, \pm 1/2, \pm 1, \ldots$. Actually, there is a subtle technical point in the quantization of h: the elementary proof that, for $SU(2)$, j_z is quantized is of an algebraic nature. One defines $\lambda_m = \langle j, m{+}1|(J_x{+}iJ_y)|jm\rangle$ and, using the commutation relations between the three J_i, one finds a recursion relation $|\lambda_{m-1}|^2 - |\lambda_m|^2 = 2m$. The condition that this recursion relation does not produce a negative $|\lambda_m|^2$ provides the quantization of $m = j_z$.[13] In the case of the little group of massless particles we do not have J_x, J_y at our disposal, but only the single $SO(2)$ generator J_z and therefore this algebraic proof does not go through. There is however a topological proof, based on the fact that the universal covering of the Lorentz group is $SL(2, C)$; this is a double covering, therefore any Lorentz rotation by 4π is the same

[13] See any book on quantum mechanics, e.g. L. Schiff, Quantum Mechanics, third edition, McGraw-Hill, New York 1968, eq. (27.23).

as the identity matrix. A detailed discussion can be found in Weinberg (1995), pages 86–90.

This analysis shows that *massless particles have only one degree of freedom, and are characterized by the value h of their helicity.* On a state of helicity h, a $U(1)$ rotation of the little group is represented by

$$U(\theta) = \exp\{-ih\theta\}\,. \tag{2.131}$$

From the point of view of the representations of the Poincaré group, a massless particle with helicity $+h$ and a massless particle with helicity $-h$ are logically two different species of particles, since they belong to two different representations of the Poincaré group. However, the helicity is the projection of the angular momentum along the direction of motion, so it can be written as

$$h = \hat{\mathbf{p}} \cdot \mathbf{J} \tag{2.132}$$

where $\hat{\mathbf{p}}$ is the unit vector in the direction of propagation. We see from eq. (2.132) that the helicity is a pseudoscalar, i.e. it changes sign under parity. If the interaction conserves parity, to each particle of helicity h there must correspond another particle with helicity $-h$, and these two helicity states must enter into the theory in a symmetric way. Since the electromagnetic interaction conserves parity, it is more natural to define the photon as a representation of the Poincaré group *and* of parity, i.e. to assemble together the two states of helicity $h = \pm 1$. The two states $h = \pm 1$ are then referred to as left-handed ($h = -1$) and right-handed ($h = +1$) photons.

Similarly the two states with helicity $h = \pm 2$ that mediate the gravitational interaction are better considered as two polarization states of the same particle, the graviton:

Photon:	$m^2 = 0$, two polarization states $h = \pm 1$.
Graviton:	$m^2 = 0$, two polarization states $h = \pm 2$.

On the contrary, neutrinos have only weak interactions (apart from the much smaller gravitational interaction), which do not conserve parity, and the two states with helicity $h = \pm 1/2$ are given different names: neutrino is reserved for $h = -1/2$, and antineutrino for $h = +1/2$.

Summary of chapter

In this chapter we have introduced a number of mathematical tools that will greatly simplify our construction of classical and quantum field theories in the next chapters. We recall some important points.

- Lie group, Lie algebras and their representations have been discussed in Section 2.1. They are central concepts in modern theoretical physics, independently of our applications to the Lorentz and Poincaré group. Basically, Lie groups are the correct language for describing continuous symmetries.

- The Lorentz group is generated by rotations and boosts, and its algebra is given in eqs. (2.27)–(2.29). We have discussed its tensor representations in Section 2.4 and its spinorial representations in Section 2.5. This leads in particular to the introduction of Weyl spinors, eq. (2.56); Dirac spinors are obtained assembling a left-handed and a right-handed Weyl spinor, and are a representation of Lorentz and of parity transformations.
- Fields are functions of the coordinates with well-defined transformation properties under Poincaré transformations. Depending on their transformation properties under the Lorentz group, we have scalar fields, Weyl fields, Dirac fields, vector fields, etc.
- The study of the representations of the Poincaré group using as base space the Hilbert space of one-particle states leads to massive particles, characterized by the spin j and having $2j+1$ degrees of freedom, and massless particles, which have one degree of freedom and a definite helicity h. For the photon and for the graviton, parity considerations suggest assembling the two states with helicity $h = \pm 1$ (for the photon) and $h = \pm 2$ (for the graviton) into a single particle.

Further reading

- For Weyl and Dirac spinors see Ramond (1990), Chapter 1 and Peskin and Schroeder (1995), Chapter 3. Observe that our definitions of ψ_L and ψ_R are inverted compared to Ramond (but agree with Peskin and Schroeder). In particular, for us the boost generator on ψ_L is $+i\sigma/2$ while for Ramond is $-i\sigma/2$ and as a consequence for us the four-vector made with left-handed spinors is $\xi_L^\dagger\bar{\sigma}^\mu\psi_L$, see eq. (2.69), while for Ramond it is $\xi_L^\dagger\sigma^\mu\psi_L$ (the fact that we both say that ψ_L belongs to the $(1/2, 0)$ representation is due to the fact that we write (j_-, j_+) while Ramond writes (j_+, j_-)). In the next chapter we will see that with our definition ψ_L has helicity $-1/2$ (and therefore with the definition of Ramond it has $h = +1/2$).
- A very clear book on Lie groups for physicists is Georgi (1999). The second edition contains many improvements of the already 'classical' first edition. For a more geometrical approach to Lie groups, see, e.g., Nakahara (1990), Section 5.6. An advanced book is J. Fuchs and C. Schweigert, *Symmetries, Lie Algebras and Representations*, Cambridge University Press 1997.
- For the representations of the Poincaré group see Sections 2.4 and 2.5 of Weinberg (1995).

Exercises

(2.1) Consider a massive particle moving with velocity $v = \tanh\eta$. Show that, if E is the energy of the particle and p its momentum along the propagation direction, then

$$\eta = \frac{1}{2}\log\frac{E+p}{E-p}, \qquad (2.133)$$