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"A modern introduction to QFT"

the operators are just the fields, so in the Heisenberg representation the quantum fields depend both on $\mathbf{x}$ and t while in the Schrödinger representation they depend only on $\mathbf{x}$. The Heisenberg representation is therefore more natural from the point of view of Lorentz covariance.

Given a state $|a\rangle(t)$ in the Schrödinger representation, in the Heisenberg picture we define the state $|a\rangle_{\rm H}$ as $|a\rangle_{\rm H} = e^{iHt}|a\rangle(t)$. If A is an operator in the Schrödinger representation, the corresponding Heisenberg operator A_H is defined as $A_{\rm H}(t) = e^{iHt}Ae^{-iHt}$. Since $|a\rangle(t)$ evolves with e^{-iHt}, and A is time-independent, by definition in the Heisenberg picture the states $|a\rangle_{\rm H}$ are independent of t while the operators A_H evolve with time. Writing $|a\rangle_{\rm H} = e^{iHt}|a\rangle(t)$ at time $t = T_i$ and recalling that we denoted $|a\rangle(T_i)$ simply as $|a\rangle$, we can write

$$|a, T_i\rangle_{\rm H} = e^{iHT_i}|a\rangle\,. \tag{5.13}$$

Note that, even if it is time-independent, the Heisenberg state $|a\rangle_{\rm H}$ carries a label T_i which was implicit in the definition of $|a\rangle$, and therefore we have denoted it as $|a, T_i\rangle_{\rm H}$. This label tells us of what Heisenberg operator the state $|a, T_i\rangle_{\rm H}$ is an eigenvector. For instance, suppose that in the Schrödinger representation the state $|x_0\rangle$, at $t = t_0$, is an eigenvector of the position operator $\hat{x}$, and let $\hat{x}_{\rm H}(t) = e^{iHt}\hat{x}e^{-iHt}$. Then the state $|x_0, t_0\rangle_{\rm H} = e^{iHt}|x_0\rangle(t)$ is an eigenvector of the Heisenberg position operator $\hat{x}_{\rm H}(t_0)$ but it is not an eigenvector of the operator $\hat{x}_{\rm H}(t_1)$ with $t_1 \neq t_0$.

Similarly to eq. (5.13) (and omitting hereafter the subscript "H" on states in the Heisenberg representation), we have

$$|b, T_f\rangle = e^{iHT_f}|b\rangle\,, \tag{5.14}$$

and in terms of the states in the Heisenberg picture the matrix element (5.5) is written as

$$\langle b|S|a\rangle = \langle b, T_f|a, T_i\rangle\,. \tag{5.15}$$

5.2 The LSZ reduction formula

Consider a generic S-matrix element written in the Heisenberg picture,

$$\langle \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n; T_f|\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_m; T_i\rangle\,. \tag{5.16}$$

It is understood that at the end of the computation $T_f \to +\infty$ and $T_i \to -\infty$. For notational simplicity we consider a single species of neutral scalar particle, so the states are labeled just by their momenta, but all our considerations can be generalized to particles with spin. Our first step will be to relate this matrix element to the expectation value of some operator on the vacuum state.

We begin by observing that the expansion of a *free* real scalar field in terms of creation and annihilation operators, eq. (4.2), can be inverted to give

$$(2E_{\mathbf{k}})^{1/2}a_{\mathbf{k}} = i\int d^3x\, e^{ikx}\, \overleftrightarrow{\partial_0}\phi_{\rm free}\,, \tag{5.17}$$

$$(2E_{\mathbf{k}})^{1/2}a_{\mathbf{k}}^{\dagger} = -i\int d^3x\, e^{-ikx}\, \overleftrightarrow{\partial_0}\phi_{\rm free}\,, \tag{5.18}$$

as one easily verifies substituting eq. (4.2) in the above equations and performing the integration over d^3x. Note that in eqs. (5.17) and (5.18) the integrands are time-dependent but the integrals are independent of t. We have denoted the field by $\phi_{\rm free}$ to stress that eqs. (5.17) and (5.18) hold only if the field is free. When the field is not free, it cannot be expanded in terms of creation and annihilation operators as in eq. (4.2), and eqs. (5.17) and (5.18) do not hold.

However, as $t \to -\infty$ we intuitively expect that the theory reduces to a free theory, since all incoming particles are infinitely far apart and, if the interaction decreases sufficiently fast with the distance, there will be no difference between a free and an interacting theory.[1] These intuitive considerations are formalized by the hypothesis that, as $t \to -\infty$,

[1] The most important example where the interaction does not decrease at large distances is the interaction of quarks in QCD. As a consequence, quarks are not seen as free particles (they are "confined" inside hadrons), and the free particles seen at $t \to \pm\infty$ are rather the hadrons. We will discuss in Problem 8.2 how to proceed in these cases.

$$\phi(x) \to Z^{1/2}\phi_{\rm in}(x)\,, \tag{5.19}$$

where $\phi_{\rm in}(x)$ is a free field and Z is a c-number, known as wave function renormalization. We will discuss later the physical meaning of Z, and how to compute it. Similarly we assume that, as $t \to +\infty$,

$$\phi(x) \to Z^{1/2}\phi_{\rm out}(x)\,, \tag{5.20}$$

with $\phi_{\rm out}$ again a free field, and the same constant Z. The limits in eqs. (5.19) and (5.20) must be understood in the weak sense, i.e. they are assumed to hold not as operator equations, but only when we take matrix elements.[2]

[2] Using a technique known as Källen–Lehmann representation (see Weinberg (1995), Section 10.7) one can show that eq. (5.19) cannot hold as an operator equation, since otherwise one would find that $Z = 1$ and that ϕ is a free field; see, e.g., Itzykson and Zuber (1980), Section 5.1.2.

We now consider eq. (5.18) with $\phi_{\rm in}$ playing the role of the free field $\phi_{\rm free}$. As we observed above, the integrand in eq. (5.18) is time-dependent, but the result of the integration is independent of t. We can therefore perform it at $t \to -\infty$, and use eq. (5.19) to write

$$\begin{aligned}(2E_{\mathbf{k}})^{1/2}a_{\mathbf{k}}^{\dagger,({\rm in})} &= -i\int_{t\to-\infty} d^3x\, e^{-ikx}\, \overleftrightarrow{\partial_0}\phi_{\rm in}\\ &= -iZ^{-1/2}\lim_{t\to-\infty}\int d^3x\, e^{-ikx}\, \overleftrightarrow{\partial_0}\phi\,,\end{aligned} \tag{5.21}$$

where the superscript "in" means that the operator $a_{\mathbf{k}}^{\dagger}$ acts on the space of initial states at $T_i = -\infty$. Similarly, we define creation operators acting on the final states as

$$\begin{aligned}(2E_{\mathbf{k}})^{1/2}a_{\mathbf{k}}^{\dagger,({\rm out})} &= -i\int_{t\to+\infty} d^3x\, e^{-ikx}\, \overleftrightarrow{\partial_0}\phi_{\rm out}\\ &= -iZ^{-1/2}\lim_{t\to+\infty}\int d^3x\, e^{-ikx}\, \overleftrightarrow{\partial_0}\phi\,.\end{aligned} \tag{5.22}$$

Observe that in eqs. (5.21) and (5.22) the final integral depends on time, since it is performed with ϕ rather than with a free field; $a_{\mathbf{k}}^{\dagger,(\text{in})}$ is defined taking the limit $t \to -\infty$ of this integral while $a_{\mathbf{k}}^{\dagger,(\text{out})}$ is defined taking the limit $t \to +\infty$, and the relation between in and out creation operators is non-trivial. Recalling our normalization (4.10) for one-particle states, we see that we can eliminate the particle with momentum $\mathbf{k}_1$ from the initial state writing

$$\begin{aligned}
&\langle \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n; T_f | \mathbf{k}_1, \mathbf{k}_2, \ldots \mathbf{k}_m; T_i \rangle \\
&= (2E_{\mathbf{k}_1})^{1/2} \langle \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n; T_f | a_{\mathbf{k}_1}^{\dagger,(\text{in})} | \mathbf{k}_2, \ldots, \mathbf{k}_m; T_i \rangle \qquad (5.23) \\
&= -iZ^{-1/2} \lim_{t \to -\infty} \int d^3x \, e^{-ik_1 x} \langle \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n; T_f | \overleftrightarrow{\partial_0} \phi | \mathbf{k}_2, \ldots, \mathbf{k}_m; T_i \rangle \, .
\end{aligned}$$

The idea is to iterate the process removing all particles from the initial and final states. We perform the computation in detail.

First of all, eq. (5.23) can be written in an explicitly covariant form. We use the fact that, for any integrable function $f(t, \mathbf{x})$, we have the identity

The reader uninterested in the derivation can just take note of the definition of time-ordered product in eq. (5.32) and then can jump directly to eq. (5.40).

$$\left(\lim_{t \to +\infty} - \lim_{t \to -\infty} \right) \int d^3x \, f(t, \mathbf{x}) = \int_{-\infty}^{\infty} dt \, \frac{\partial}{\partial t} \int d^3x \, f(t, \mathbf{x}) \, . \qquad (5.24)$$

Applying this identity to the function $f(t, \mathbf{x}) = -iZ^{-1/2} e^{-ikx} \overleftrightarrow{\partial_0} \phi$ and using eqs. (5.21) and (5.22) we find

$$(2E_{\mathbf{k}})^{1/2} (a_{\mathbf{k}}^{\dagger,(\text{in})} - a_{\mathbf{k}}^{\dagger,(\text{out})}) = iZ^{-1/2} \int d^4x \, \partial_0 (e^{-ikx} \overleftrightarrow{\partial_0} \phi) \, . \qquad (5.25)$$

The integral in this equation can be written in a covariant form observing that

$$\begin{aligned}
\int d^4x \, \partial_0 (e^{-ikx} \overleftrightarrow{\partial_0} \phi) &= \int d^4x \, \partial_0 (e^{-ikx} \partial_0 \phi - \phi \, \partial_0 e^{-ikx}) \\
&= \int d^4x \, (e^{-ikx} \partial_0^2 \phi - \phi \partial_0^2 e^{-ikx}) \\
&= \int d^4x \, \left[e^{-ikx} \partial_0^2 \phi - \phi (\nabla^2 - m^2) \, e^{-ikx} \right] \, , \qquad (5.26)
\end{aligned}$$

where in the last line we used the fact that $k^2 = m^2$, since k^μ is the four-momentum of an initial or final particle with mass m, and therefore $\partial_0^2 e^{-ikx} = (\nabla^2 - m^2) e^{-ikx}$. It is understood that our initial and final particle states, which we have written simply as states with definite momentum, i.e. plane waves, will be convoluted to form wave packets, so at each given time they are localized in space. This means that we can integrate ∇^2 twice by parts (while ∂_0 cannot be integrated by parts, since ϕ is not localized in time), and we find

$$(2E_{\mathbf{k}})^{1/2} (a_{\mathbf{k}}^{\dagger,(\text{in})} - a_{\mathbf{k}}^{\dagger,(\text{out})}) = iZ^{-1/2} \int d^4x \, e^{-ikx} (\Box + m^2) \phi(x) \, . \qquad (5.27)$$

Therefore

$$\begin{aligned}
&(2E_{\mathbf{k}_1})^{1/2} \langle \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n; T_f | a_{\mathbf{k}_1}^{\dagger,(\text{in})} - a_{\mathbf{k}_1}^{\dagger,(\text{out})} | \mathbf{k}_2, \ldots, \mathbf{k}_m; T_i \rangle \qquad (5.28) \\
&= iZ^{-1/2} \int d^4x \, e^{-ik_1 x} (\Box + m^2) \langle \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n; T_f | \phi(x) | \mathbf{k}_2, \ldots, \mathbf{k}_m; T_i \rangle \, .
\end{aligned}$$

The operator $a^{\dagger,(\mathrm{out})}_{\mathbf{k}_1}$ acts on the state to its left, destroying an out particle with momentum $\mathbf{k}_1$. We assume that none of the initial momenta $\mathbf{p}_j$ coincides with a final momentum $\mathbf{k}_i$. This eliminates processes in which one of the particles behaves as a "spectator" and does not interact with the other particles.[3] Then $a^{\dagger,(\mathrm{out})}_{\mathbf{k}_1}$ acting on the state on its left gives zero, because the particle that it would annihilate is absent, and the left-hand side of eq. (5.28) coincides with the expression that appears in eq. (5.23).

[3] In the language of Feynman diagrams that we will explain below, this means that we can restrict to connected diagrams.

The conclusion is that we can remove the particle with momentum $\mathbf{k}_1$ from the initial state, at the price of inserting the operator

$$iZ^{-1/2}\int d^4x\, e^{-ik_1x}(\Box+m^2)\phi(x) \tag{5.29}$$

in the matrix element, i.e.

$$\begin{aligned}&\langle \mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n;T_f|\mathbf{k}_1,\mathbf{k}_2,\ldots,\mathbf{k}_m;T_i\rangle \\ &= iZ^{-1/2}\int d^4x\, e^{-ik_1x}(\Box+m^2)\langle \mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n;T_f|\phi(x)|\mathbf{k}_2,\ldots,\mathbf{k}_m;T_i\rangle\,.\end{aligned} \tag{5.30}$$

Now we would like to iterate the procedure, eliminating all initial and final particles and remaining with the vacuum expectation value of some combination of fields. For instance, we next eliminate the final particle with momentum $\mathbf{p}_1$. Following the same strategy adopted before, we write

$$\begin{aligned}&\langle \mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n;T_f|\phi(x)|\mathbf{k}_2,\ldots,\mathbf{k}_m;T_i\rangle \\ &= (2E_{\mathbf{p}_1})^{1/2}\langle \mathbf{p}_2,\ldots,\mathbf{p}_n;T_f|a^{(\mathrm{out})}_{\mathbf{p}_1}\phi(x)|\mathbf{k}_2,\ldots,\mathbf{k}_m;T_i\rangle\,.\end{aligned} \tag{5.31}$$

We now define the *time-ordered product*, or simply the T-product, of two fields as follows,

$$T\{\phi(y)\phi(x)\} = \begin{cases}\phi(y)\phi(x) & y^0>x^0\\ \phi(x)\phi(y) & y^0<x^0\end{cases} \tag{5.32}$$

or

$$T\{\phi(y)\phi(x)\} = \theta(y^0-x^0)\phi(y)\phi(x)+\theta(x^0-y^0)\phi(x)\phi(y)\,, \tag{5.33}$$

where $\theta(x^0)$ is the step function: $\theta(x^0)=1$ if $x^0>0$ and $\theta(x^0)=0$ if $x^0<0$. Taking the hermitian conjugate of eq. (5.21) we see that $a^{(\mathrm{in})}_{\mathbf{p}_1}$ is constructed in terms of $\phi(y)$ with $y^0\to-\infty$, and therefore

$$T\{a^{(\mathrm{in})}_{\mathbf{p}_1}\phi(x)\} = \phi(x)a^{(\mathrm{in})}_{\mathbf{p}_1}\,. \tag{5.34}$$

Similarly, $a^{(\mathrm{out})}_{\mathbf{p}_1}$ is constructed in terms of $\phi(y)$ with $y^0\to+\infty$ and

$$T\{a^{(\mathrm{out})}_{\mathbf{p}_1}\phi(x)\} = a^{(\mathrm{out})}_{\mathbf{p}_1}\phi(x)\,. \tag{5.35}$$

We can use this to write the right-hand side of eq. (5.31) as

$$(2E_{\mathbf{p}_1})^{1/2}\langle \mathbf{p}_2,\ldots,\mathbf{p}_n;T_f|T\{(a^{(\mathrm{out})}_{\mathbf{p}_1}-a^{(\mathrm{in})}_{\mathbf{p}_1})\phi(x)\}|\mathbf{k}_2,\ldots,\mathbf{k}_m;T_i\rangle\,. \tag{5.36}$$

In fact, the first term in the T-product is the same as the original expression in eq. (5.31), while the second gives zero since we have seen that $T\{a^{(\mathrm{in})}_{\mathbf{p}_1}\phi(x)\}=\phi(x)a^{(\mathrm{in})}_{\mathbf{p}_1}$ and then $a^{(\mathrm{in})}_{\mathbf{p}_1}$ annihilates the state on its right (recall that we are assuming that the final momenta $\mathbf{p}_j$ are different from any of the initial momenta $\mathbf{k}_i$).

The advantage of the form (5.36) is that the combination $a^{(\text{out})}_{\mathbf{p}_1} - a^{(\text{in})}_{\mathbf{p}_1}$ is given in terms of a covariant expression involving the ϕ field, which is just the hermitian conjugate of eq. (5.27),

$$(2E_{\mathbf{p}_1})^{1/2}(a^{(\text{out})}_{\mathbf{p}_1} - a^{(\text{in})}_{\mathbf{p}_1}) = iZ^{-1/2}\int d^4y\, e^{ip_1y}(\Box_y + m^2)\phi(y)\,. \tag{5.37}$$

Therefore

$$\begin{aligned}&\langle \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n; T_f|\phi(x)|\mathbf{k}_2, \ldots, \mathbf{k}_m; T_i\rangle \\ &= iZ^{-1/2}\int d^4y\, e^{ip_1y}(\Box_y + m^2)\langle \mathbf{p}_2, \ldots, \mathbf{p}_n; T_f|T\{\phi(y)\phi(x)\}|\mathbf{k}_2, \ldots, \mathbf{k}_m; T_i\rangle\,,\end{aligned} \tag{5.38}$$

where $\Box_y = \frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y_\mu}$.[4] Putting together eqs. (5.30) and (5.38) we find the result of eliminating the particles with momenta $\mathbf{k}_1$ and $\mathbf{p}_1$,

$$\begin{aligned}&\langle \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n; T_f|\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_m; T_i\rangle \\ &= (iZ^{-1/2})^2\int d^4x\, e^{-ik_1x}(\Box_x + m^2)\int d^4y\, e^{+ip_1y}(\Box_y + m^2) \\ &\quad\times\langle \mathbf{p}_2, \ldots, \mathbf{p}_n; T_f|T\{\phi(y)\phi(x)\}|\mathbf{k}_2, \ldots, \mathbf{k}_m; T_i\rangle\,.\end{aligned} \tag{5.39}$$

The procedure can now be iterated in a straightforward way, and the result is

$$\begin{aligned}&\langle \mathbf{p}_1, \ldots, \mathbf{p}_n; T_f|\mathbf{k}_1, \ldots, \mathbf{k}_m; T_i\rangle \\ &= (iZ^{-1/2})^{n+m}\int \prod_{i=1}^m d^4x_i \prod_{j=1}^n d^4y_j\, \exp(i\sum_{j=1}^n p_jy_j - i\sum_{i=1}^m k_ix_i) \\ &\quad\times(\Box_{x_1} + m^2)\ldots(\Box_{y_n} + m^2)\langle 0|T\{\phi(x_1)\ldots\phi(y_n)\}|0\rangle\,,\end{aligned} \tag{5.40}$$

where the T-product $T\{\phi(x_1)\ldots\phi(y_n)\}$ by definition orders the $n+m$ fields $\phi(x_1), \ldots, \phi(y_m)$ according to decreasing times, so that larger times are leftmost. The vacuum at $t = \pm\infty$ is the perturbative vacuum, i.e. the vacuum used in the construction of the Fock space of the free theory.[5]

As we explained in Section 5.1, $\langle \mathbf{p}_1 \ldots \mathbf{p}_n; T_f|\mathbf{k}_1 \ldots \mathbf{k}_m; T_i\rangle$ is the matrix element in the Heisenberg representation. In the Schrödinger representation we write instead

$$\langle \mathbf{p}_1 \ldots \mathbf{p}_n|S|\mathbf{k}_1 \ldots \mathbf{k}_m\rangle\,. \tag{5.41}$$

We have also defined the operator T from $S = 1 + iT$. Since in eq. (5.40) we restricted to the situation in which no initial and final momenta coincide, the matrix element of the identity operator between these states vanishes, and we have actually computed the matrix element of iT, i.e. of the non-trivial part of the evolution operator,

$$\begin{aligned}&\langle \mathbf{p}_1 \ldots \mathbf{p}_n|iT|\mathbf{k}_1 \ldots \mathbf{k}_m\rangle \\ &= (iZ^{-1/2})^{n+m}\int \prod_{i=1}^m d^4x_i \prod_{j=1}^n d^4y_j\, \exp(i\sum_{j=1}^n p_jy_j - i\sum_{i=1}^m k_ix_i) \\ &\quad\times(\Box_{x_1} + m^2)\ldots(\Box_{y_n} + m^2)\langle 0|T\{\phi(x_1)\ldots\phi(y_n)\}|0\rangle\,.\end{aligned} \tag{5.42}$$

[4] A very technical remark: writing eq. (5.38) we have extracted $\Box_y$ from the T-product; strictly speaking this is not correct, because $\partial/\partial y^0$ does not commute with the theta function that enters in the definition of the T-product, since $\partial_x\theta(x) = \delta(x)$. However, a simple calculation shows that the additional term is proportional to $\delta(x^0 - y^0)[\partial_0\phi(y), \phi(x)] \sim \delta^{(4)}(x - y)$, and the inclusion of this Dirac delta (and of its derivatives, coming from acting on it with the $\Box_x$ operators present in the LSZ formula) modifies the final result for the LSZ formula, eq. (5.46), by the addition of terms which are polynomial in the four-momenta. Since however both the left-hand side and the right-hand side of eq. (5.46) are pole-like in the four-momenta, i.e. proportional to factors $1/(p^2 - m^2)$, the addition of a regular term is irrelevant when we go on mass shell, i.e. when we set $p^2 = m^2$; see the discussion below eq. (5.46).

[5] Observe that initial one-particle states are defined from $|\mathbf{k}\rangle = (2E_{\mathbf{k}})^{1/2}a_{\mathbf{k}}^{\dagger,(\text{in})}|0\rangle$ and final states from $|\mathbf{k}\rangle = (2E_{\mathbf{k}})^{1/2}a_{\mathbf{k}}^{\dagger,(\text{out})}|0\rangle$, with the same state $|0\rangle$ in both cases, including its phase.

We now define the N-point Green's function

$$G(x_1,\dots,x_N) = \langle 0|T\{\phi(x_1)\dots\phi(x_N)\}|0\rangle\,. \tag{5.43}$$

In terms of its Fourier transform $\tilde{G}$, we have

$$G(x_1,\dots,x_N) = \int \prod_{i=1}^{N} \frac{d^4k_i}{(2\pi)^4}\, e^{-i\sum_{i=1}^{N} x_i k_i}\, \tilde{G}(k_1,\dots,k_N)\,. \tag{5.44}$$

Using

$$\begin{aligned}(\Box_{x_j} + m^2)&G(x_1,\dots,x_N)\\ &= -\int \prod_{i=1}^{N} \frac{d^4k_i}{(2\pi)^4}\,(k_j^2 - m^2) e^{-i\sum_{i=1}^{N} x_i k_i} \tilde{G}(k_1,\dots,k_N)\,,\end{aligned} \tag{5.45}$$

eq. (5.42) can be rewritten as

$$\begin{aligned}&\prod_{i=1}^{m}\int d^4 x_i e^{-ik_i x_i} \prod_{j=1}^{n}\int d^4 y_j e^{+ip_j y_j}\\ &\quad\times\langle 0|T\{\phi(x_1)\dots\phi(x_m)\phi(y_1)\dots\phi(y_n)\}|0\rangle\\ &= \left(\prod_{i=1}^{m}\frac{i\sqrt{Z}}{k_i^2 - m^2}\right)\left(\prod_{j=1}^{n}\frac{i\sqrt{Z}}{p_j^2 - m^2}\right)\langle \mathbf{p}_1\dots\mathbf{p}_n|iT|\mathbf{k}_1\dots\mathbf{k}_m\rangle\,.\end{aligned} \tag{5.46}$$

This is the Lehmann–Symanzik–Zimmermann (LSZ) reduction formula. It is important to understand the meaning of the factors $k_i^2 - m^2$ and $p_j^2 - m^2$ in the denominator. Of course for a physical particle with four-momentum p^μ we have $p^2 - m^2 = 0$ (which is often expressed saying that the particle is "on mass shell"). The meaning of these factors is that we must first compute the left-hand side of eq. (5.46) working off mass shell, i.e. without using any relation between p_0^2 and $\mathbf{p}^2$. In the limit in which we send the particles on mass shell, the left-hand side develops poles of the form $1/(k_i^2 - m^2)$ for each incoming particle and $1/(p_j^2 - m^2)$ for each outgoing particle. These factors cancel the same pole factors which appear explicitly on the right-hand side, and we remain with an equation between quantities that are finite when the particles are on mass shell.

We have therefore succeeded in relating the scattering amplitude to the vacuum expectation value of a time-ordered product of fields. In the next section we will see how the latter can be computed order by order in perturbation theory.

5.3 Setting up the perturbative expansion

At the classical level, the field $\phi(x)$ satisfies a complicated non-linear equation of motion, determined by the full Lagrangian $L_0 + L_{\text{int}}$ which