PHY 5667 : Quantum Field Theory A, Fall 2017

October 5^{th}, 2017
Assignment # 3
(due Thursday October 19^{th}, 2017)

1. This problem teaches you how to calculate $\langle\Omega|T\phi(x_1)\dots\phi(x_n)|\Omega\rangle$ using canonical quantization. We will discussed some of its steps in class, on Tuesday Oct. 10. It could be beneficial for you to have a look at the problem before the lesson.

Consider a quantum system of fields with Hamiltonian density $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ where $\mathcal{H}_0 = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$, and $\mathcal{H}_1$ is a function of $\pi(0,\mathbf{x})$ and $\phi(0,\mathbf{x})$ and their spatial derivatives. (It should be chosen to preserve Lorentz invariance, but we will not be concerned with this issue here.) Let us define $|\Omega\rangle$ to be the vacuum state of H such that $H|\Omega\rangle = 0$, and let us instead denote by $|0\rangle$ the vacuum state of H_0 such that $H_0|0\rangle = 0$. The Heisenberg-picture field $\phi(x)$ is

$$\phi(t,\mathbf{x}) \equiv e^{iHt}\phi(0,\mathbf{x})e^{-iHt} \ ,$$

while the *interaction-representation* or *interaction-picture* field is defined as

$$\phi_I(t,\mathbf{x}) \equiv e^{iH_0t}\phi(0,\mathbf{x})e^{-iH_0t} \ .$$

1.a Show that $\phi_I(x)$ obeys the Klein-Gordon equation, and hence is a free field.

1.b Show that $\phi(x) = U^\dagger(t)\phi_I(x)U(t)$, where $U(t) \equiv e^{iH_0t}e^{-iHt}$ is unitary.

1.c Show that $U(t)$ obeys the differential equation $i\frac{d}{dt}U(t) = H_I(t)U(t)$, where $H_I(t) = e^{iH_0t}H_1e^{-iH_0t}$ is the interaction Hamiltonian in the interaction representation, and the boundary condition is $U(0) = 1$.

1.d Show that if $\mathcal{H}_1$ is specified by a particular function of the fields $\pi(0,\mathbf{x})$ and $\phi(0,\mathbf{x})$, show that $\mathcal{H}_I(t)$ is given by the same function of the interaction-picture fields $\pi_I(t,\mathbf{x})$ and $\phi_I(t,\mathbf{x})$.

1.e Show that, for $t > 0$,

$$U(t) = T\exp\left[-i\int_0^t dt' H_I(t')\right]$$

obeys the differential equation and boundary conditions of part (**1.c**). What is the comparable expression for $t < 0$? (*Hint*: you might need to define a new ordering symbol.)

1.f Define $U(t_1,t_2) \equiv U(t_2)U^\dagger(t_1)$ and show that for $t_2 > t_1$

$$U(t_2,t_1) = T\exp\left[-i\int_{t_1}^{t_2} dt' H_I(t')\right] \ .$$

What is the comparable expression for $t_2 < t_1$?

1.g For any time ordering show that $U(t_3,t_1) = U(t_3,t_2)U(t_2,t_1)$ and that $U^\dagger(t_1,t_2) = U(t_2,t_1)$.

1.h Show that

$$\phi(x_n)\dots\phi(x_1) = U^\dagger(t_n,0)\phi_I(x_n)U(t_n,t_{n-1})\phi_I(x_{n-1})\dots U(t_2,t_1)\phi_I(x_1)U(t_1,0)\ .$$

1.i Show that $U^\dagger(t_n,0) = U^\dagger(\infty,0)U(\infty,t_n)$ and also that $U(t_1,0) = U(t_1,-\infty)U(-\infty,0)$.

1.j Replace H_0 with $(1-i\epsilon)H_0$ ($\epsilon > 0$, $\epsilon \ll 1$) (notice that this is related to the analytic continuation used in defining the Feynman propagator), and show that $\langle\Omega|U^\dagger(\infty,0) = \langle\Omega|0\rangle\langle 0|$ and that $U(-\infty,0)|\Omega\rangle = |0\rangle\langle 0|\Omega\rangle$.

1.k Show that

$$\begin{aligned}\langle\Omega|\phi(x_n)\dots\phi(x_1)|\Omega\rangle &= \langle 0|U(\infty,t_n)\phi_I(x_n)U(t_n,t_{n-1})\phi_I(x_{n-1})\dots \\ &\quad \dots U(t_2,t_1)\phi_I(x_1)U(t_1,-\infty)|0\rangle|\langle 0|\Omega\rangle|^2\ .\end{aligned}$$

1.l Show that

$$\langle\Omega|T\phi(x_n)\dots\phi(x_1)|\Omega\rangle = \langle 0|T\phi_I(x_n)\dots\phi_I(x_1)e^{-i\int d^4x\mathcal{H}_I(x)}|0\rangle|\langle 0|\Omega\rangle|^2\ .$$

1.m Show that

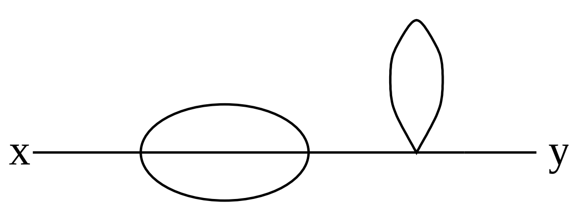
$$|\langle 0|\Omega\rangle|^2 = \frac{1}{\langle 0|Te^{-i\int d^4x\mathcal{H}_I(x)}|0\rangle}\ .$$

Thus we have

$$\langle\Omega|T\phi(x_n)\dots\phi(x_1)|\Omega\rangle = \frac{\langle 0|T\phi_I(x_n)\dots\phi_I(x_1)e^{-i\int d^4x\mathcal{H}_I(x)}|0\rangle}{\langle 0|Te^{-i\int d^4x\mathcal{H}_I(x)}|0\rangle}\ .$$

We can now expand the exponential on the right-hand side of the last equation in (**1.m**), and use the formalism of a free-field theory to compute the resulting correlation functions. This is the core of the perturbative approach to any interacting field theory.

2. The following Feynman diagram:



represents a contribution to $\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle$. Explain which term of the perturbative expansion of $\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle$ corresponds to this diagram and write the corresponding analytical contribution both in position space and in momentum space.

3. Consider a real scalar field theory with $\mathcal{H}_{\text{int}} = \frac{\lambda}{4!}\phi^4$.

3.a) Derive the tree-level vertex interaction that defines the vertex Feynman rule for this theory in momentum space.

3.b) Given a generic diagram with E external legs, V vertices, and P propagators, what is the relation among E, V, and P for both connected and non-connected diagrams?

3.c) Draw all the fully connected diagrams with $1 \leq E \leq 4$ and $0 \leq V \leq 2$, and find their symmetry factors.

4. Repeat the same exercise for the case of a complex scalar field with $\mathcal{H}_{\text{int}} = \frac{\lambda}{4}(\phi^\dagger \phi)^2$. Remember that there are now two kinds of particles (which we can think as positively and negatively charged), and that your rules must have a way to distinguish among them. *Hint*: it seems like you need two kinds of *arrows* to represent the flow of momentum and charge separately. Try to find a more elegant approach that only needs one kind of arrows.