PHY 5667 : Quantum Field Theory A, Fall 2017

November 2^{nd}, 2017

Assignment # 5

(due Thursday November 16^{th}, 2017)

1. By definition, a representation of $SU(2)$ is a set of $n \times n$ matrices σ_1, σ_2, σ_3 that satisfy the algebra $[\sigma_i, \sigma_j] = i\epsilon_{ijk}\sigma_k$. For $n = 2$ these are the Pauli matrices. In general, you can build any finite dimensional irreducible representation of $SU(2)$ through the following steps that you are asked to complete:

1.a) In such generic representation, you can diagonalize σ_3. Its eigenvectors are n complex vectors V_j such that $\sigma_3 V_j = \lambda_j V_j$. Define $\sigma_\pm = \sigma_1 \pm i\sigma_2$ and show that $\sigma_+ V_j$ and $\sigma_- V_j$ either vanish or are eigenstates of σ_3 with eigenvalues $(\lambda_j + 1)$ and $(\lambda_j - 1)$ respectively.

1.b) Prove that one and only one eigenstate of σ_3, $V_{\rm max}$, must satisfy $\sigma_+ V_{\rm max} = 0$. The eigenvalue $\lambda_{\rm max} = j$ of $V_{\rm max}$ is known as the *spin*. Similarly, prove that there is one and only one eigenvector $V_{\rm min}$ such that $\sigma_- V_{\rm min} = 0$.

1.c) Since there are a finite number of eigenvectors, $V_{\rm min} = (\sigma_-)^N V_{\rm max}$ for some integer N. Prove that $N = 2j$ such that $n = 2j + 1$.

1.d) Construct explicitly the $n = 5$ (five-dimensional) representation of $SU(2)$.

2. Let us write the infinitesimal form of a Lorentz transformation in the *vector representation* as

$$\Lambda^\rho_\sigma = \delta^\rho_\sigma - \frac{i}{2}\delta\omega_{\mu\nu}(J_V^{\mu\nu})^\rho_\sigma \ ,$$

where

$$(J_V^{\mu\nu})^\rho_\sigma = i\left(g^{\mu\rho}\delta^\nu_\sigma - g^{\nu\rho}\delta^\mu_\sigma\right) \ ,$$

are matrices in the vector representation of the Lorentz generators ($g^{\mu\nu}$ denotes the metric tensor in Minkowski space and δ^ν_μ is the Kronecker δ in four dimensions).

2.a) Write Λ^μ_ν for a rotation by an angle θ about the x axis, and show that,

$$\Lambda = \exp\left(-i\theta J_V^{23}\right) \ .$$

2.b) Write Λ^μ_ν for a boost by rapidity η in the z direction, and show that,

$$\Lambda = \exp\left(i\eta J_V^{30}\right) \ .$$

3. Under a Lorentz transformation (Λ) Dirac (and Majorana) fields transform as,

$$\psi'(x) = D(\Lambda)\psi(\Lambda^{-1}x) \ ,$$

where, denoting by $S^{\mu\nu}$ the generators of Lorentz transformations in the Dirac spinor representation, $D(\Lambda)$ for an infinitesimal transformation can be written as,

$$D(\Lambda) = 1 + \frac{i}{2}\delta\omega_{\mu\nu}S^{\mu\nu} \ .$$

3.a) Find the form of the generators $S^{\mu\nu}$.

3.b) Find in this representation the explicit form of a finite rotation by an angle θ about the z axis.

3.c) Find in this representation the explicit form of a finite boost by rapidity η in the z direction.

4. Consider Weyl spinors $\xi_{L,R}$ and $\psi_{L,R}$, and prove that $\xi_L^\dagger \bar{\sigma}^\mu \psi_L$ and $\xi_R^\dagger \sigma^\mu \psi_R$ transform as Lorentz 4-vectors.

5. Problem 3.4 of Peskin and Schroeder's book.