PHY 5667 : Quantum Field Theory A, Fall 2018

August 30^{th}, 2018
Assignment # 1
(due Thursday September 6^{th}, 2018)

1. It is often possible to derive a field theory as the limit of a discrete system. Perhaps the simplest example is an infinite system of aligned point masses, m, separated by massless springs of spring constant k and equilibrium length a. This model can be used to (approximately) describe both the longitudinal vibrations of an elastic rod and the transverse oscillations of a stretched string. Let η_i be the displacement from equilibrium of the ith point mass. Derive the exact Lagrangian and the Euler-Lagrange equations for this system. Then consider the limit

$$m, a \to 0 \ , \ k \to \infty \ , \ \mu = m/a \text{ and } Y = ka \text{ fixed} \ .$$

Replacing η_i by a smooth function $\eta(x,t)$, show that in this limit the Lagrangian may be written in the density form

$$L = \int dx \frac{1}{2} \left[\mu \left(\frac{\partial \eta}{\partial t} \right)^2 - Y \left(\frac{\partial \eta}{\partial x} \right)^2 \right]$$

and write down the corresponding Euler-Lagrange equation. This is a notable equation (which one?) that you have obtained as the continuous limit of a discrete system of coupled harmonic oscillators!

2. The electromagnetic field may be specified by a vector $A^\mu(\mathbf{x}, t)$, in terms of which the Lagrange density of the field is

$$\mathcal{L}(x) = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \ ,$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ .$$

Derive the Lagrange equations for this system, and express them in terms of the free-space field strengths $\mathbf{E} = -\partial_0 \mathbf{A} - \nabla A_0$ and $\mathbf{B} = \nabla \times \mathbf{A}$. How many of Maxwell's equations does this give, and why are the others also satisfied?