PHY 5667 : Quantum Field Theory A, Fall 2018

September 6^{th}, 2018
Assignment # 2
(due Thursday September 20^{th}, 2018)

1. Problem 2.2 of Schwartz's book.

2. Problem 2.4 of Schwartz's book.

3. Problem 2.6 of Schwartz's book.

4. Let us write the infinitesimal form of a Lorentz transformation in the *vector representation* as

$$\Lambda^\rho_\sigma = \delta^\rho_\sigma - \frac{i}{2}\delta\omega_{\mu\nu}(J_V^{\mu\nu})^\rho_\sigma \ ,$$

where

$$(J_V^{\mu\nu})^\rho_\sigma = i\left(g^{\mu\rho}\delta^\nu_\sigma - g^{\nu\rho}\delta^\mu_\sigma\right) \ ,$$

are matrices in the vector representation of the Lorentz generators ($g^{\mu\nu}$ denotes the metric tensor in Minkowski space and δ^ν_μ is the Kronecker δ in four dimensions).

4.a) Write Λ^μ_ν for an infinitesimal rotation by an angle θ about the x axis, and show that the corresponding finite rotation is given by,

$$\Lambda = \exp\left(-i\theta J_V^{23}\right) \ .$$

4.b) Write Λ^μ_ν for an infinitesimal boost by rapidity η in the z direction, and show that the corresponding finite boost is given by,

$$\Lambda = \exp\left(i\eta J_V^{30}\right) \ .$$

5. Consider a generic system of fields $\phi_i(x)$. Using Noether's theorem show that:

5.a The invariance of the action S,

$$S = \int d^4x\, \mathcal{L}(\phi_i(x), \partial_\mu\phi_i(x), x) \ ,$$

under an infinitesimal space-time translation,

$$x'^\mu = x^\mu + \epsilon^\mu \ ,$$

implies the existence of four conserved currents,

$$T^\mu_\nu = -\mathcal{L}\,\delta^\mu_\nu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\partial_\nu\phi_i \ ,$$

known as *energy-momentum tensor* of the system of fields, and four conserved charges,

$$Q^i = \int T^{0i} d^3x \quad \text{and} \quad Q^0 = \int T^{00} d^3x \ ,$$

which can be interpreted as the three components of the *momentum* carried by the system of fields and its *energy*.

5.b The invariance of the action S under an infinitesimal Lorentz transformation of both the fields and the coordinates,

$$\begin{aligned} x'^{\mu} &= \Lambda^{\mu}_{\nu} x^{\nu} \simeq x^{\mu} + \sum_{k=1}^{6} \alpha_k X^{\mu}_k + O(\alpha^2) \ , \\ \phi'_i(x') &= L_{ij}(\Lambda)\phi_j(x) \simeq \phi_i(x) + \sum_{k=1}^{6} \alpha_k A_{ij,k}\phi_j(x) + O(\alpha^2) \ , \end{aligned}$$

where $L(\Lambda)$ denote the representation of the Lorentz group on the space of the fields $\phi_i(x)$, implies the existence of six conserved currents (Noether's currents) of the form,

$$M^{\mu}_{\rho\sigma} = T^{\mu}_{\rho} x_{\sigma} - T^{\mu}_{\sigma} x_{\rho} - \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi_i(x))} A_{ij,\rho\sigma}\phi_j(x) \ ,$$

where $T^{\mu\nu}$ is the energy-momentum tensor associated to the system of fields, and six conserved charges. How are the charges defined? How can you interpret the components of $M^{\mu}_{\rho\sigma}$ due to the transformation of the coordinates and to the trasformation of the fields respectively? Explain your reasoning.

6. Classical electromagnetism (with no sources) follows from the action,

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \ , \quad \text{where } F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \ ,$$

as you have derived in your first homework. Construct the energy-momentum tensor for this theory. Notice that the usual procedure does not result in a symmetric tensor. Show that you can remedy that using a tensor $K^{\lambda\mu\nu}$ antisymmetric in its first two indices (you will have to build a new energy-momentum tensor $\hat{T}^{\mu\nu} = T^{\mu\nu} + \ldots$ where the dots stays for a function of $K^{\lambda\mu\nu}$). In particular show that

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^{\nu} \ ,$$

leads to the standard formulae for the electromagnetic energy and momentum densities:

$$\mathcal{E} = \frac{1}{2}(E^2 + B^2) \ , \qquad \mathbf{S} = \mathbf{E} \times \mathbf{B} \ .$$