PHY 5667 : Quantum Field Theory A, Fall 2018

September 20^{th}, 2018
Assignment # 3
(due Thursday October 4^{th}, 2018)

1. Consider the theory of a complex scalar field whose Lagrangian density is given by:

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \,,$$

where complex means complex-valued. Notice that $\phi(x)$ and $\phi^*(x)$ are here considered as independent dynamical variables (instead of the real and imaginary part of $\phi(x)$).

1.a) Show that $\phi(x)$ (and $\phi^*(x)$) satisfies the Klein-Gordon equation.

1.b) Find the conjugate momenta of $\phi(x)$ and $\phi^*(x)$ and show that the Hamiltonian is:

$$H = \int d^3x \left(\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi \right) \,.$$

1.c) Define your quantization procedure in term of canonical commutation relations among the $\phi(x)$, $\phi^*(x)$, $\pi(x)$, and $\pi^*(x)$ operators. Introduce *annihilation* and *creation* operators and calculate their commutation relations. How many kinds of such operators do you need to introduce? Show that the theory contains two sets of particles of mass m.

1.d) The system admits a conserved charge Q due to the invariance of its action functional under a $U(1)$ transformation of the fields:

$$\begin{aligned} \phi(x) &\rightarrow \phi'(x) = e^{i\alpha} \phi(x) \,, \\ \phi^*(x) &\rightarrow (\phi^*)'(x) = e^{-i\alpha} \phi^*(x) \,, \end{aligned}$$

where α is constant. Calculate Q in terms of creation and annihilation operators, and evaluate the charge of the particles of each type.

2. As a complement to our discussion in class, this problem allows you to review and complement the calculation of ${}_{\text{out}}\langle 0 | T\phi(x_1) \ldots \phi(x_n) | 0 \rangle_{\text{in}}$ for a system of interacting real scalar fields, using canonical quantization. We have discussed some of its steps in class, and will use the results in our lesson on Oct. 2^{nd}. It could be beneficial for you to have a look at the problem before that lesson.

Consider a quantum system of fields with Hamiltonian density $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ where $\mathcal{H}_0 = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$, and $\mathcal{H}_1$ is a function of $\pi(0, \mathbf{x})$ and $\phi(0, \mathbf{x})$ (or equivalently $\pi(t_0, \mathbf{x})$ and $\phi(t_0, \mathbf{x})$, for a fixed time t_0) and their spatial derivatives. Let us define $|0\rangle_{\text{in,out}}$ to be the vacuum states of the system before ($T_i \rightarrow -\infty$) and after ($T_f \rightarrow +\infty$) the interaction, corresponding to a system of free fields $\phi^{\text{in}}(x)$ and $\phi^{\text{out}}(x)$ respectively. The Heisenberg-picture field $\phi(x) = \phi(t, \vec{x})$ is

$$\phi(t, \mathbf{x}) \equiv e^{iH(t-t_0)} \phi(t_0, \mathbf{x}) e^{-iH(t-t_0)} \,,$$

while the *interaction-representation* or *interaction-picture* field is defined as

$$\phi_I(t, \mathbf{x}) \equiv e^{iH_0(t-t_0)} \phi(t_0, \mathbf{x}) e^{-iH_0(t-t_0)} \,.$$

2.a Show that $\phi_I(x)$ obeys the Klein-Gordon equation, and hence is a free field.

2.b Show that $\phi(x) = U^\dagger(t,t_0)\phi_I(x)U(t,t_0)$, where $U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}$ is unitary.

2.c Show that $U(t,t_0)$ obeys the differential equation $i\frac{d}{dt}U(t,t_0) = H_I(t)U(t,t_0)$, where $H_I(t) = e^{iH_0(t-t_0)}H_1 e^{-iH_0(t-t_0)}$ is the interaction Hamiltonian in the interaction representation, and the boundary condition is $U(t_0,t_0) = 1$.

2.d Show that if the Hamiltonian density $\mathcal{H}_1$ is specified by a particular function of the fields $\pi(t_0,\mathbf{x})$ and $\phi(t_0,\mathbf{x})$, show that $\mathcal{H}_I(t)$ is given by the same function of the interaction-picture fields $\pi_I(t,\mathbf{x})$ and $\phi_I(t,\mathbf{x})$.

2.e Show that, for $t > t_0$,

$$U(t) = T\exp\left[-i\int_{t_0}^{t} dt' H_I(t')\right]$$

obeys the differential equation and boundary conditions of part (**2.c**). What is the comparable expression for $t < t_0$?

2.f Define $U(t_2,t_1) \equiv U(t_2,t_0)U^\dagger(t_1,t_0)$ and show that for $t_2 > t_1$

$$U(t_2,t_1) = T\exp\left[-i\int_{t_1}^{t_2} dt' H_I(t')\right] .$$

What is the comparable expression for $t_2 < t_1$?

2.g For any time ordering show that $U(t_3,t_1) = U(t_3,t_2)U(t_2,t_1)$ and that $U^\dagger(t_1,t_2) = U(t_2,t_1)$.

2.h Show that, given a time ordering, e.g. $x_1^0 > \ldots > x_n^0$, one has:

$$\phi(x_1)\ldots\phi(x_n) = U^\dagger(t_1,t_0)\phi_I(x_1)U(t_1,t_2)\phi_I(x_2)\ldots U(t_{n-1},t_n)\phi_I(x_n)U(t_n,t_0) .$$

2.i Show that, for any t such that $t \gg t_1 > \ldots > t_n \gg -t$, $U^\dagger(t_1,t_0) = U^\dagger(t,t_0)U(t,t_1)$ and also that $U(t_n,t_0) = U(t_n,-t)U(-t,t_0)$.

2.j Using the previous results, show that, setting $t_0 = -t$ and letting $t \to \infty$ one gets:

$$\begin{aligned} {}_{\text{out}}\langle 0|\phi(x_1)\ldots\phi(x_n)|0\rangle_{\text{in}} &= {}_{\text{out}}\langle 0|U^\dagger(\infty,-\infty)U(\infty,t_1)\phi_I(x_1)U(t_1,t_2)\phi_I(x_2)\ldots \\ &\quad \ldots U(t_{n-1},t_n)\phi_I(x_n)U(t_n,-\infty)|0\rangle_{\text{in}} . \end{aligned}$$

2.k The vacuum states $|0>_{\text{in}}$ and $|0>_{\text{out}}$ are both vacuum states of the free theory, according to what we assumed at $T_i \to -\infty$ to $T_f \to +\infty$. Hence the two states are the same modulus at most a phase since states differing by a phase are equivalent in quantum mechanics. At the same time, if the vacuum state is stable, the time evolution from $T_i \to -\infty$ to $T_f \to +\infty$ should reproduce the same state modulus a phase. We can than write that:

$$e^{i\alpha}|0\rangle = U(\infty,-\infty)|0\rangle ,$$

where the distinction between *in* and *out* vacuum states is superfluous and can be dropped. We can than use the previous relation to derive that:

$$e^{i\alpha} = \langle 0|Te^{-i\int d^4x\mathcal{H}_I(x)}|0\rangle .$$

2.l Using all previous results show that:

$$
{}_{\rm out}\langle 0|T\phi(x_1)\ldots\phi(x_n)|0\rangle_{\rm in} = \frac{\langle 0|T\phi_I(x_1)\ldots\phi_I(x_n)e^{-i\int d^4x\mathcal{H}_I(x)}|0\rangle}{\langle 0|Te^{-i\int d^4x\mathcal{H}_I(x)}|0\rangle} .
$$

We can now expand the exponential on the right-hand side of the last equation in (**1.m**), and use the formalism of a free-field theory to compute the resulting correlation functions. This is the core of the perturbative approach to any interacting field theory.