PHY 5667 : Quantum Field Theory A, Fall 2018

October 23^{rd}, 2018
Assignment # 5
(due Tuesday November 6^{th}, 2018)

1. **Problem 10.2 of Schwartz's book**.
   By definition, a representation of $SU(2)$ is a set of $n \times n$ matrices σ_1, σ_2, σ_3 that satisfy the algebra $[\sigma_i, \sigma_j] = i\epsilon_{ijk}\sigma_k$. For $n = 2$ these are the Pauli matrices. In general, you can build any finite dimensional irreducible representation of $SU(2)$ through the following steps that you are asked to complete:

   1.a) In such generic representation, you can diagonalize σ_3. Its eigenvectors are n complex vectors V_j such that $\sigma_3 V_j = \lambda_j V_j$. Define $\sigma_\pm = \sigma_1 \pm i\sigma_2$ and show that $\sigma_+ V_j$ and $\sigma_- V_j$ either vanish or are eigenstates of σ_3 with eigenvalues $(\lambda_j + 1)$ and $(\lambda_j - 1)$ respectively.

   1.b) Prove that one and only one eigenstate of σ_3, $V_{\max}$, must satisfy $\sigma_+ V_{\max} = 0$. The eigenvalue $\lambda_{\max} = j$ of $V_{\max}$ is known as the *spin*. Similarly, prove that there is one and only one eigenvector $V_{\min}$ such that $\sigma_- V_{\min} = 0$.

   1.c) Since there are a finite number of eigenvectors, $V_{\min} = (\sigma_-)^N V_{\max}$ for some integer N. Prove that $N = 2j$ such that $n = 2j + 1$.

   1.d) Construct explicitly the $n = 5$ (five-dimensional) representation of $SU(2)$.

2. Under a Lorentz transformation (Λ) Dirac (and Majorana) fields transform as,

$$\psi'(x) = D(\Lambda)\psi(\Lambda^{-1}x) \ ,$$

   where, denoting by $S^{\mu\nu}$ the generators of Lorentz transformations in the Dirac spinor representation, $D(\Lambda)$ for an infinitesimal transformation can be written as,

$$D(\Lambda) = 1 + \frac{i}{2}\delta\omega_{\mu\nu}S^{\mu\nu} \ .$$

   2.a) Find the form of the generators $S^{\mu\nu}$.

   2.b) Find in this representation the explicit form of a finite rotation by an angle θ about the z axis.

   2.c) Find in this representation the explicit form of a finite boost by rapidity η in the z direction.

3. Consider Weyl spinors $\xi_{L,R}$ and $\psi_{L,R}$, and prove that $\xi_L^\dagger \bar{\sigma}^\mu \psi_L$ and $\xi_R^\dagger \sigma^\mu \psi_R$ transform as Lorentz 4-vectors.

4. **Extra credit**. Problem 4.3 of Peskin and Schroeder's book.