"How to calculate $\beta(\lambda)$, $\gamma_m(\lambda)$, $\gamma(\lambda)$ for a $\lambda\phi^4$ theory using dimensional regularization + MS".

(P.2)

Consider as in illustration the $\lambda\phi^4$ case.

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi_0\,\partial_\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4$$

$$= \frac{1}{2}Z\,\partial^\mu\phi\,\partial_\mu\phi - \frac{1}{2}m_0^2 Z\phi^2 - \frac{\lambda_0}{4!}Z^2\phi^4$$

$$= \frac{1}{2}Z\,\partial^\mu\phi\,\partial_\mu\phi - \frac{1}{2}m^2 Z_0\,\phi^2 - \frac{\lambda}{4!}Z_1\phi^4$$

$$= \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 +$$

$$\frac{1}{2}(Z-1)\partial^\mu\phi\,\partial_\mu\phi - \frac{1}{2}(Z_0-1)m^2\phi^2 - \frac{\lambda}{4!}(Z_1-1)\phi^4$$

where :

$$\lambda_0 Z^2 = \lambda Z_1 \longrightarrow \boxed{\lambda_0\mu^{-2\epsilon}Z^2 = \lambda Z_1}$$

$$m_0^2 Z = m^2 Z_0$$

so :

$$\lambda = \lambda_0 Z_1^{-1} Z^2 \mu^{-2\epsilon}$$

s.t.

$$\beta(\lambda) \Rightarrow \beta(\lambda,\epsilon) = \mu\frac{\partial}{\partial\mu}\lambda = -2\epsilon\lambda + \lambda\left(\mu\frac{\partial}{\partial\mu}\overbrace{Z_1^{-1}Z^2}^{Z_\lambda^{-1}}\right) \xrightarrow[\epsilon\to 0]{} \beta(\lambda)$$

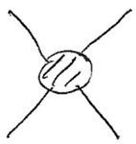
$$\gamma(\lambda) \Rightarrow \gamma(\lambda,\epsilon) = \frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z \xrightarrow[\epsilon\to 0]{} \gamma(\lambda)$$

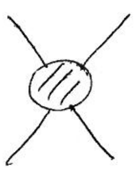
$$\gamma_m(\lambda) \Rightarrow \gamma_m(\lambda,\epsilon) = \frac{\mu}{m}\frac{\partial m}{\partial\mu} = \mu\frac{\partial}{\partial\mu}(\overbrace{Z Z_0^{-1}}^{Z_m}) \xrightarrow[\epsilon\to 0]{} \gamma_m(\lambda)$$

to recall the notation } At each order $\longrightarrow$ $\lambda \simeq \lambda_0\mu^{-2\epsilon}$ after having taken derivatives let's briefly remind ourselves of how we should proceed in renormalizing the $\lambda\phi^4$ theory, regularizing it in dimensional regularization and subtracting the UV-div using minimal subtraction (MS).

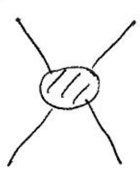
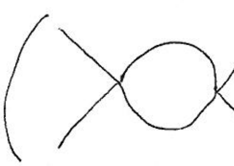
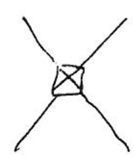
two divergent primitive amplitudes:


$D = 2$


$D = 4$

(1)  $\equiv -i\lambda$ (at no renorm. scale, since MS and since $D=0$)

at $O(\lambda^2)$:

 $=$  $+$ ( + 2 more) $+$ 

↓ $O(\lambda)$ ↓ $O(\lambda^2)$ CT, of the form $-i\lambda\,\delta Z_1 = -i\lambda(Z_1 - 1)$

the first one-loop diagram contributes:

$$A_1(s) = +i\lambda \frac{\lambda}{16\pi^2}\frac{1}{\epsilon} + \underbrace{\text{finite}}_{\downarrow}$$

$$-i\lambda \frac{\lambda}{(4\pi)^2}\frac{1}{2}\left\{\int_0^1 dx \ln\left(\frac{m^2 - s x(1-x)}{\mu^2}\right) + \gamma - \ln(4\pi)\right\}$$

the three diagrams are therefore:

$$A_1 = i\lambda \frac{\lambda}{16\pi^2}\frac{3}{\epsilon} + \left(A_1(s) + A_1(t) + A_1(u)\right)_{\text{finite}}$$

MS : subtract the pole part only :

$$(Z_1 - 1)_{MS} = \frac{\lambda}{16\pi^2} \frac{3}{\epsilon}$$

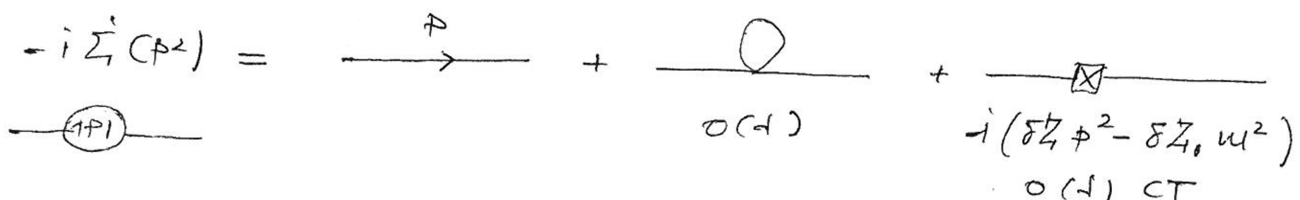
$\overline{MS}$: subtract along also some "omnipresent" finite terms (that originate, along with poles either from $\Gamma(\epsilon)$ or from phase space

$$(Z_1 - 1)_{\overline{MS}} = \frac{\lambda}{16\pi^2} \frac{3}{2} \left(\frac{2}{\epsilon} - \gamma + \ln 4\pi \right)$$

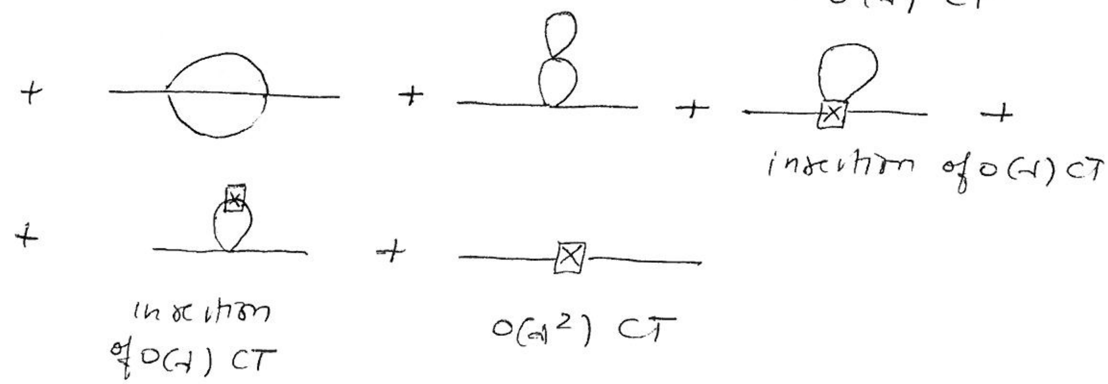
see how this appear in the expression of A_1.

(2)  $= \dfrac{i Z}{p^2 - m^2 + \Sigma(p^2)}$

$-i\Sigma(p^2) =$



we haven't done explicitely this part of the calculation, but it's in the notes.



imposing MS, i.e. subtracting only the pole parts proportional to m^2 and p^2 one gets :

$$(Z_0 - 1) = \frac{\lambda}{16\pi^2} \frac{1}{\epsilon} + \frac{\lambda^2}{(16\pi^2)^2} \left(\frac{2}{\epsilon^2} - \frac{1}{2} \frac{1}{\epsilon} \right)$$

$$(Z - 1) = - \frac{\lambda^2}{(16\pi^2)^2} \frac{1}{12} \frac{1}{\epsilon}$$

Now we have all the ingredients to start our RG analysis, and calculate, for the particular case of the $\lambda\phi^4$ theory:

$$\beta(\lambda,\epsilon) = \mu \frac{d\lambda}{d\mu} \xrightarrow[\epsilon\to 0]{} \beta(\lambda)$$

$$\gamma(\lambda,\epsilon) = \frac{1}{2}\mu \frac{d}{d\mu} \ln Z_\phi \xrightarrow[\epsilon\to 0]{} \gamma(\lambda)$$

$$\gamma_m(\lambda,\epsilon) = \frac{\mu}{m}\frac{dm}{d\mu} = \frac{1}{2}\frac{1}{m^2}\mu\frac{dm^2}{d\mu} \xrightarrow[\epsilon\to 0]{} \gamma_m(\lambda)$$

(1) $$\beta(\lambda,\epsilon) = \mu\frac{d\lambda}{d\mu} = \mu\frac{d}{d\mu}\left(\lambda_0\mu^{-\epsilon}\underbrace{Z_\phi^2 Z_1^{-1}}_{Z_\lambda^{-1}}\right)$$

$$\lambda_0 Z_\phi^2 = \mu^\epsilon \lambda Z_1$$

$$\lambda = \lambda_0 \mu^{-\epsilon} Z_\phi^2 Z_1^{-1}$$

$$= -\epsilon\lambda_0\mu^{-\epsilon}Z_\lambda^{-1} - \mu^{-\epsilon}\lambda_0 Z_\lambda^{-2}\mu\frac{dZ_\lambda}{d\mu}$$

$$= -\epsilon\lambda - \lambda Z_\lambda^{-1}\mu\frac{dZ_\lambda}{d\mu}$$

now, we know that:

$$Z_\lambda = \left[1 + \sum_k \frac{Z_\lambda^{(k)}}{\epsilon^k}\right]$$

while $\beta(\lambda,\epsilon)$ is regular for $\epsilon\to 0$, i.e. $\beta(\lambda,\epsilon) = \beta^{(\lambda)}\epsilon^{\lambda}$ then we can write the previous relation as follows:

$$\beta(\lambda,\epsilon) = -\epsilon\lambda - \lambda Z_\lambda^{-1}\mu\frac{d\lambda}{d\mu}\frac{dZ_\lambda}{d\lambda} =$$

$$= -\epsilon\lambda - \lambda Z_\lambda^{-1}\beta(\lambda,\epsilon)\, dZ_\lambda/d\lambda$$

$$\beta(\alpha,\epsilon)\, Z_\alpha + \epsilon\alpha\, Z_\alpha + \alpha\, \beta(\alpha,\epsilon)\, \frac{dZ_\alpha}{d\alpha} = 0$$

$$\downarrow$$

$$\beta(\alpha,\epsilon)\left[1 + \sum_k \frac{Z_\alpha^{(k)}}{\epsilon^k}\right] + \epsilon\alpha\left[1 + \sum_k \frac{Z_\alpha^{(k)}}{\epsilon^k}\right] +$$

$$\alpha\, \beta(\alpha,\epsilon) \sum_k \frac{dZ_\alpha^{(k)}}{d\alpha}\,\frac{1}{\epsilon^k} = 0 \qquad (*) \rightarrow$$

equating the residua at the poles on both sides one gets the following relations:

$O(\epsilon) + O(1)$:

$$\beta(\alpha,\epsilon) + \epsilon\alpha - \cancel{\epsilon}\,\alpha^2 \frac{dZ_\alpha^{(1)}}{d\alpha}\,\frac{1}{\cancel{\epsilon}} = 0$$

$$\beta(\alpha,\epsilon) = -\epsilon\alpha + \beta(\alpha) \qquad \text{where } \beta(\alpha) = \alpha^2 \frac{dZ_\alpha^{(1)}}{d\alpha}$$

and for higher order poles (they have to cancel):

$$-\alpha^2 \frac{dZ_\alpha^{(k)}}{d\alpha} + \alpha\,\beta(\alpha)\, dZ_\alpha^{(k)} + \underbrace{(\beta(\alpha,\epsilon) + \epsilon\alpha)}_{\beta(\alpha)}\, Z_\alpha^{(k)} = 0$$

$$-\alpha^2 \frac{dZ_\alpha^{(k+1)}}{d\alpha} + \beta(\alpha)\, d(\alpha Z_\alpha^{(k)}) = 0$$

(*)

$$\beta(\lambda,\epsilon)\left[Z_\lambda+\lambda\frac{dZ_\lambda}{d\lambda}\right]=-\epsilon\lambda Z_\lambda$$

$$\beta(\lambda,\epsilon)\cdot\left[1+\sum_k\frac{Z_\lambda^{(k)}}{\epsilon^k}+\lambda\sum_k\frac{dZ_\lambda^{(k)}}{d\lambda}\cdot\frac{1}{\epsilon^k}\right]=$$

$$-\epsilon\lambda\left[1+\sum_k\frac{Z_\lambda^{(k)}}{\epsilon^k}\right]$$

$$\beta(\lambda,\epsilon)=-\epsilon\lambda+\beta(\lambda)\qquad\text{(ansatz)}$$

$$(-\epsilon\lambda+\beta(\lambda))\left[1+\sum_k\frac{Z_\lambda^{(k)}}{\epsilon^k}+\lambda\cdot\sum_k\frac{dZ_\lambda^{(k)}}{d\lambda}\frac{1}{\epsilon^k}\right]=-\epsilon\lambda\left[1+\sum_k\frac{Z_\lambda^{(k)}}{\epsilon^k}\right]$$

$$-\epsilon\lambda\cdot\lambda\sum_k\frac{dZ_\lambda^{(k)}}{d\lambda}\frac{1}{\epsilon^k}+\beta(\lambda)\left[1+\sum_k\frac{Z_\lambda^{(k)}}{\epsilon^k}+\lambda\sum_k\frac{dZ_\lambda^{(k)}}{d\lambda}\frac{1}{\epsilon^k}\right]=$$

$$-\lambda^2\frac{dZ_\lambda^{(1)}}{d\lambda}+\beta(\lambda)=0\qquad\text{(no pole)}$$

$$-\lambda^2\frac{dZ_\lambda^{(k+1)}}{d\lambda}+\beta(\lambda)\left[Z_\lambda^{(k)}+\lambda\frac{dZ_\lambda^{(k)}}{d\lambda}\right]=0\qquad\text{(k-th pole)}$$

$$\hookrightarrow\quad-\lambda^2\frac{dZ_\lambda^{(k+1)}}{d\lambda}=\beta(\lambda)\frac{d}{d\lambda}\left(\lambda Z_\lambda^{(k)}\right)$$

(2) $$\gamma(\lambda) = \frac{1}{2}\mu\frac{d\ln Z}{d\mu} = \frac{1}{2}\frac{1}{Z}\mu\frac{dZ}{d\mu} =$$

$$= \frac{1}{2}\frac{1}{Z}\beta(\lambda,\epsilon)\frac{dZ}{d\lambda}$$

$$Z = 1 + \sum_k{}' \frac{Z^{(k)}}{\epsilon^k}$$

$$\gamma(\lambda) = \frac{1}{2}\left(1+\sum_k \frac{Z^{(k)}}{\epsilon^k}\right)^{-1}\left(-\epsilon\lambda+\beta(\lambda)\right)\sum_k \frac{dZ^{(k)}}{d\lambda}\frac{1}{\epsilon^k}$$

$$= -\frac{1}{2}\lambda\frac{dZ^{(1)}}{d\lambda} + \text{pole terms that has to cancel}$$

(3) $$\gamma_m(\lambda) = \frac{1}{2}\frac{1}{m^2}\mu\frac{dm^2}{d\mu} = \frac{1}{2}\frac{1}{m_0^2 Z_m^{-1}}\mu\frac{d}{d\mu}\left(m_0^2 Z_m^{-1}\right)$$

$$\downarrow$$

$$(Z^{-1}Z_0)^{-1}$$

$$= -\frac{1}{2} Z_m \frac{1}{Z_m^2}\mu\frac{d}{d\mu}Z_m = -\frac{1}{2}\frac{1}{Z_m}\mu\frac{dZ_m}{d\mu}$$

$$= -\frac{1}{2}\frac{1}{Z_m}\beta(\lambda,\epsilon)\frac{dZ_m}{d\lambda}$$

$$\rightarrow \quad Z_m = 1 + \sum_k{}' \frac{Z_m^{(k)}}{\epsilon^k}$$

s.t.

$$\gamma_m(\lambda) = -\frac{1}{2}\left(1+\sum_k \frac{Z_m^{(k)}}{\epsilon^k}\right)^{-1}\left(-\epsilon\lambda+\beta(\lambda)\right)\sum_k \frac{dZ_m^{(k)}}{d\lambda}\frac{1}{\epsilon^k}$$

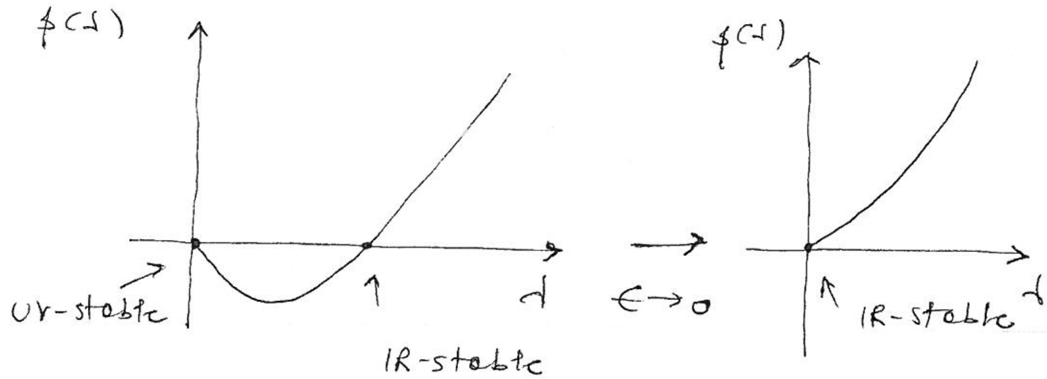
$$= +\frac{1}{2}\lambda\frac{dZ_m^{(1)}}{d\lambda} + \text{pole terms that has to cancel}$$

So, if we now use the expressions for Z_0, Z_1 and Z that we have summarized initially for the $\lambda\phi^4$ theory, we obtain.

$$\beta(\lambda,\epsilon) = -\epsilon\lambda + \lambda^2 \frac{d Z_\lambda^{(1)}}{d\lambda}$$

$$Z_\lambda = Z^{-2} Z_1 = \left(1 + \frac{\lambda}{16\pi^2}\frac{3}{\epsilon}\right)\left(1 - \frac{\lambda^2}{(16\pi^2)^2}\frac{1}{12}\frac{1}{\epsilon}\right)^{-2}$$

$$\beta(\lambda,\epsilon) = -\epsilon\lambda + \frac{\lambda^2}{16\pi^2}\cdot 3 + O(\lambda^3)$$



$d \neq 4$ $\qquad$ $d = 4$

$$\gamma(\lambda) = -\frac{1}{2}\lambda\frac{dZ^{(1)}}{d\lambda} = \frac{\lambda^2}{(16\pi^2)^2}\cdot\frac{1}{12} + O(\lambda^3)$$

$$\gamma_m(\lambda) = -\frac{1}{2}\lambda\frac{dZ_m^{(1)}}{d\lambda} = \frac{1}{2}\frac{\lambda}{16\pi^2} - \frac{5}{12}\frac{\lambda^2}{(16\pi^2)^2} + O(\lambda^3)$$

$$Z_m = Z_0 Z^{-1} = 1 + \frac{1}{\epsilon}\left[\frac{\lambda}{16\pi^2} - \frac{5}{12}\frac{\lambda^2}{(16\pi^2)^2}\right] + \frac{1}{\epsilon^2}\frac{2\lambda^2}{(16\pi^2)^2}$$

... and QED

For QED, the situation is even simpler. Thanks to gauge invariance, we have that:

$$\alpha_0 = \mu^{\epsilon} Z_\alpha \alpha \qquad \text{where} \quad Z_\alpha = Z_3^{-1} \qquad \alpha = \frac{e^2}{4\pi}$$

where:

$$Z_\alpha = 1 + \frac{\alpha}{\pi}\frac{2}{3}\frac{1}{\epsilon} + \left(\frac{\alpha}{\pi}\right)^2 \frac{b_2}{\epsilon} + \dots$$

s.t.

$$\beta(\alpha) = \alpha\left[\frac{\alpha}{\pi}\frac{2}{3} + \left(\frac{\alpha}{\pi}\right)^2 b_2 + \dots\right]$$

or

$$\beta(e) = \frac{e^3}{16\pi^2}\frac{4}{3} + \frac{e^5}{(16\pi^2)^2} 8 b_2 + \dots$$

$$\left\{ \beta(\alpha) = \mu\frac{\partial\alpha}{\partial\mu} = \mu\frac{\partial}{\partial\mu}\left(\frac{e^2}{4\pi}\right) = \frac{2e}{4\pi}\mu\frac{\partial e}{\partial\mu} = \frac{2e}{4\pi}\beta(e) \right.$$

$$\left. \beta(e) = \frac{\overset{2}{\cancel{4}}\pi}{\cancel{2}e}\cdot\beta(\alpha) = \frac{2\pi}{e}\beta(\alpha) \right\}.$$

as for $\lambda\phi^4$, $\beta(e) > 0 \longrightarrow \alpha \neq 0 \; e = 0$ IR $\overbrace{\text{fixed point}}^{\text{stable}}$ not UV.

"the running coupling $\alpha(\mu)$"

Once we have $\beta = \beta(\lambda)$ or $\beta = \beta(\alpha)$ we can solve for λ or α and find the explicit expression of $\lambda(\mu)$ or $\alpha(\mu)$ order by order in perturbation theory, i.e. o.b.y.o. in α.

Let us just use one symbol, α, for both couplings. To be meant as $\alpha = \lambda$ (for $\lambda\phi^4$) and $\alpha = \frac{e^2}{4\pi}$ (for QED).

The expansion of $\beta(\alpha)$ in powers of α can be written as:

$$\beta(\alpha) = \alpha\left[\left(\frac{\alpha}{\pi}\right) b_1 + \left(\frac{\alpha}{\pi}\right)^2 b_2 + \left(\frac{\alpha}{\pi}\right)^3 b_3 + \dots\right]$$

we can then calculate $\alpha(\mu)$ from the equation:

$$t = \ln\left(\frac{\mu}{\mu_0}\right) \leftrightarrow \mu = e^{+t}\mu_0$$

$$\mu \frac{\partial \bar{\alpha}}{\partial \mu} = \beta(\bar{\alpha}) \quad \longrightarrow \quad + \frac{\partial \bar{\alpha}}{\partial t} = \beta(\bar{\alpha})$$

with $\bar{\alpha}(0, \alpha) = \alpha$

comment about the sign difference with respect to Pokorski here. (should be uninfluent because it enters the same twice).

indeed, writing:

$$\bar{\alpha}(t) = \sum_{m=1}^{\infty} a_m(t)\, \alpha^m \qquad a_1(t) = 1$$

we get:

$$\sum_{m=2}^{\infty} \frac{d a_m(t)}{dt} \alpha^m = \sum_{n=1}^{\infty} \frac{b_n}{\pi^n} \Big[\underbrace{\alpha + \sum_{m=2}^{\infty} a_m(t)\, \alpha^m\Big]^{n+1}}_{\alpha^{n+1}}$$

from which we can get the coefficients $a_m(t)$ comparing same powers of α on the left and right h.s.

α^2: $$\frac{da_2(t)}{dt} = \frac{b_1}{\pi} \rightarrow a_2(t) = \frac{b_1 t}{\pi}$$

α^3: $$\frac{da_3(t)}{dt} = \frac{b_2}{\pi} + \frac{b_1}{\pi} 2 a_2(t)$$
$$= \frac{b_2}{\pi} + \left(\frac{b_1}{\pi}\right)^2 2t$$
$$\hookrightarrow a_3(t) = \frac{b_2}{\pi} t + \left(\frac{b_1 t}{\pi}\right)^2$$

$\vdots$

α^n: $$a_n(t) = \left(\frac{b_1}{\pi} t\right)^{n-1} + O(t^{n-2})$$

such that, we can rethink the expansion of $\alpha(t)$ in towers (or powers) of (αt) instead of α, as follows:

$$\alpha(t) = \alpha + \sum_{m=2}^{\infty} a_m(t)\, \alpha^m$$
$$= \alpha \left[1 + \sum_{n=1}^{\infty} \left(\frac{b_1}{\pi} \alpha t\right)^n + O\left(\alpha^n t^{n-1}\right)\right]$$

. etc.

leading logarithms — next-to-leading logarithms

etc.

to calculate $\alpha(t)$ at L.L. level, means to only keep the first tower of logs, and this is obtainable by calculating just the b_1 coefficient in the β-function, i.e. with a lowest order (or 1-loop) calculation to calculate $\alpha(t)$ at NLL level we need also b_2, and this implies to get the β-function at 2-loop level, and so on so far.

<u>example</u>:

$$t = \ln\left(\frac{\mu}{\mu_0}\right)$$

$$\alpha(\mu_0) = \alpha$$

$$\beta(\alpha) = \alpha \cdot \frac{\alpha}{\pi} b_1$$

$$\frac{d\alpha}{dt} = \frac{b_1}{\pi}\alpha^2$$

$$\frac{d\alpha}{\alpha^2} = \frac{b_1}{\pi} dt$$

$$-\frac{1}{\alpha}\Big|_{\mu_0}^{\mu} = \frac{b_1}{\pi} t\Big|_{\mu_0 \to t=0}^{\mu \to t}$$

$$-\frac{1}{\alpha(\mu)} + \frac{1}{\alpha(\mu_0)} = \frac{b_1}{\pi} t$$

$$\frac{1}{\alpha(\mu)} = \frac{1}{\alpha} - \frac{b_1}{\pi} t \longrightarrow \alpha(\mu) = \frac{\alpha}{1 - \frac{b_1}{\pi}\alpha t}$$

$$(1-x)^{-1} = 1 + x + x^2 + \dots + x^n + \dots$$

this is equivalent to:

$$\alpha(\mu) = \alpha\left[1 + \sum_{n=1}^{\infty}\left(\frac{b_1}{\pi}\alpha t\right)^n\right]$$

Leading-logarithmic form of $\alpha(\mu)$.

So:

in LO calculations, or 1-loop calculations, need α_{LL}
in NLO " , or 2-loop calculations, need α_{NLL}