Introduction to the "Renormalization Group" ideas and formalism

① Why "renormalization", and why "group"?
We have seen how in any QFT the presence of UV-diverg. can be removed at the level of physical observables.
This is true at all orders for theories that we deem "renormalizable", and it is still true in "non-renormalizable" (or "effective") field theories order by order in the $(p/\Lambda)^n$ expansion ($p \to$ scale of momenta of the theory, $\Lambda \to \Lambda_{UV}$ high-energy scale of new physics beyond the one considered).

The way in which the renormalization program is implemented depends on what we choose as regularization and renormalization scheme.
It is recognized by most theorists that the most convenient and formally consistent regularization method is Dimensional Regularization ($d=4 \to d<4$ for UV).
The choice of Renormalization scheme is more debatable, and depends on the theory, as we will discuss later.

↓

(the theoretical predictions of) physical observables do not depend on the renormalization scheme used to calculate them.

↳ this is the "invariance" to which the "group" term refers to.
the "renormalization" part is obvious from the above discussion.

This way of introducing the idea of "renormalization group" is often referred to as "Continuum RG", as opposed to the "Wilsonian RG" approach, although the two are very much related.

Indeed, the Wilsonian RG approach can be summarized in the statement that "in a finite theory, with a UV cutoff Λ, the theory at a scale $E \ll \Lambda$ is independent of Λ".

↳ changing Λ changes the couplings of the theory so that observables remain the same.

↓

We will discuss the consequent formalism more in detail later on, but we can already see how the fundamental idea is always that of controlling the dependence on / influence of the UV cutoff (UV physics).

Since many scales in a theory can serve as UV cutoff, depending on the phenomenon to be studied, both continuum and Wilsonian RG approaches give the possibility of relating the same theory at different scales.

(2) "Invariance of observables under changes in how they are calculated"

Let's consider two prototype renormalization schemes:

- OS scheme → CT defined by imposing that the renormalized masses correspond to poles in the ren. propagators and the renormalized couplings to the couplings extracted in specific exp. measurements.

- $\overline{MS}$ → CT defined to subtract just the pole parts of vertex interactions and propagators.

the continuum RG invariance can be expressed by the fact that a given observable $\mathcal{O}$ is the same when calculated at all perturbative orders in either scheme, i.e.:

$$\mathcal{O}_{OS}\left(\alpha, m, \ln\frac{s}{m^2}, \ldots\right) = \mathcal{O}_{\overline{MS}}\left(\alpha(\mu), m(\mu), \ln\frac{s}{\mu^2}, \ldots\right)$$

where, for simplicity, we have specified the discussion to a QED observable, with just one fermionic species of mass m.

$\sqrt{s} \to$ center of mass energy

$\ldots \to$ other possible dependencies on s and other invariants.

Comments

(a) each scheme inherently depends on some mass scales. these can be physical mass scales (such as m, or $q^2=0$, or $q^2=m^2$ at which the renormalization conditions are given, such as in OS; or unphysical scales such as the "sliding scale" μ in $\overline{MS}$)

Notice that: μ is introduced via dimensional regularization, so it should affect both OS and $\overline{MS}$. But, in OS the CT also depend on μ, and the μ-dep cancels at the level of renormalized quantities. this does not happen in $\overline{MS}$ since the CT are μ-independent.

(b) $\overline{MS}$ depends on a "sliding" or "arbitrary" scale and this can be used to derive evolution equations that determine the scale-dependence of the components of a given theoretical prediction and used to improve them-

(e) While $\{\alpha, m\}$ are well-defined physical objects (they are observables on their own!), $\{\alpha(\mu), m(\mu)\}$ are theoretical objects, defined by a perturbative expansion in α!

↓

(e.1) $$\alpha_0 = Z_\alpha^{\overline{MS}}(\mu)\,\alpha_{\overline{MS}}(\mu) = Z_\alpha^{OS}\,\alpha_{OS} \quad\rightarrow\quad \alpha_{\overline{MS}}(\mu) = \frac{Z_\alpha^{OS}}{Z_\alpha^{\overline{MS}}}\,\alpha_{OS}$$

$$Z_\alpha = Z_3^{-1} \quad\rightarrow\quad \alpha_{\overline{MS}}(\mu) = \left(\frac{Z_3^{\overline{MS}}}{Z_3^{OS}}\right)\alpha_{OS}$$

$$\alpha_{\overline{MS}}(\mu) = \alpha\left[1 + \frac{\alpha}{3\pi}\ln\frac{\mu^2}{m^2} + O(\alpha^2)\right]$$

(e.2) $$m_0 = \frac{Z_m}{Z_2}\,m \quad\rightarrow\quad \left(\frac{Z_m(\mu)}{Z_2(\mu)}\right)_{\overline{MS}} m_{\overline{MS}}(\mu) = \left(\frac{Z_m}{Z_2}\right)_{OS} m_{OS}$$

$$\rightarrow\quad m_{\overline{MS}}(\mu) = m\left[1 - \frac{3\alpha}{4\pi}\left(\ln\frac{\mu^2}{m^2} + \frac{4}{3}\right) + O(\alpha^2)\right]$$

While $\{\alpha, m\}$ are fixed, $\{\alpha(\mu), m(\mu)\}$ can reabsorb part of the (large) logarithmic corrections present order by order in the calculation of σ and provide a more stable theoretical prediction.

↓

We start seeing one of the reasons why $\overline{MS}$ can be advantageous. If a theory is such that large terms of the form $\left(\alpha\ln\frac{\mu^2}{q^2}\right)^n$ ($q^2 \rightarrow$ characteristic scale) appear in the perturbative calculation at each order, it could be a big improvement to be able to take them into account and include them in theoretical predictions
↳ we will see how.

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Since $\mathcal{O}$ has to be scale independent, we can indeed write (using $\overline{MS}$) that:

$$\mu \frac{d\mathcal{O}}{d\mu} = 0 \longrightarrow \mu \frac{\partial \mathcal{O}}{\partial \alpha(\mu)} \frac{d\alpha(\mu)}{d\mu} + \mu \frac{\partial \mathcal{O}}{\partial m(\mu)} \frac{dm(\mu)}{d\mu} + \mu \frac{\partial \mathcal{O}}{\partial \mu} = 0$$

first example of RG equation (RGE).

It is customary to define:

- $\mu \frac{d\alpha(\mu)}{d\mu} \equiv \beta(\alpha(\mu)) \longrightarrow$ "β-function"
- $\mu \frac{dm(\mu)}{d\mu} = m(\mu)\, \gamma_m(\alpha(\mu)) \longrightarrow$ "mass anomalous dimension"

they govern the scale dependence of the parameters of the $\mathcal{L}_i(\alpha, m)$.

such that the previous equation become:

$$\beta(\alpha) \frac{\partial \mathcal{O}}{\partial \alpha} + m\gamma_m(\alpha) \frac{\partial \mathcal{O}}{\partial m} + \frac{\partial \mathcal{O}}{\partial \ln \mu} = 0$$

it shows how different scale-dependencies compensate each other to keep $\mathcal{O}$ scale independt.

here you can see the relation with the Wilsonian RG approach.
Parameters scale such that the prediction of the free theory remains the same.

4. Calculating $\beta(\alpha)$ and $\gamma_m(\alpha)$: see separate notes where the details of the calculation are presented both for QED and $\lambda\phi^4$.

5. "Commenting on using $\overline{MS}$ to improve theoretical predictions":

We have obtained that:

$$\alpha(\mu) = \frac{\alpha(\mu_0)}{1 + \frac{\alpha(\mu_0)}{4\pi}\beta_0 \ln\frac{\mu^2}{\mu_0^2}}$$

↓

expanding in α: $\frac{\alpha(\mu)}{\alpha(\mu_0)} \propto \left(1 + \frac{\beta_0}{4\pi n}\sum_n \left(\alpha \ln\frac{\mu^2}{\mu_0^2}\right)^n + \dots\right)$

resummation of terms of the form $\left(\alpha \ln\frac{\mu^2}{\mu_0^2}\right)^n$ → "leading logs"

These terms are typically small in QED, where there is no reason to resum them, since other terms at higher-order can be equally important. We will see, however, how in QCD these terms will become much more important ($\alpha_{QCD} \simeq 10\,\alpha_{QED}$) and having the possibility of collecting them at all orders becomes crucial.

↳ this is obtained through $\boxed{\overline{MS} + RGE}$.

and justify the use of $\overline{MS}$ in QCD (where also there is no physical correspondence to m, since quark masses are not well defined theoretically!).

[QED-wise, the most common renormalization scheme is OS since α, m are very precisely measured.]

Analogously, for $m(\mu)$ we have that:

$$m(\mu) = m(\mu_1)\left(\frac{\alpha(\mu)}{\alpha(\mu_0)}\right)^{-\gamma^0/2\beta_0}$$

$$\downarrow$$

$$\frac{1}{1+\frac{\alpha_s(\mu_0)}{4\pi}\beta_0 \ln\frac{\mu^2}{\mu_0^2}} \longrightarrow \sum_n \left(\alpha \ln\frac{\mu^2}{\mu_0}\right)^n$$

The use of the $\overline{MS}$ mass as renormalized mass allows to resum large logarithmic corrections, if necessary.

$\downarrow$

Since these logarithmic terms are relatively small in QED, while $\{\alpha, m\}$ are very accurately measured, the best scheme to be used is OS.