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DIMENSIONAL REGULARIZATION OF INFRARED DIVERGENCES

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An analysis of the application of dimensional regularization to infrared divergences in lowest order radiative corrections is presented. The main emphasis of the paper is to show explicitly how dimensional regularization can lead in some cases of considerable interest to very simple and elegant evaluations of infrared divergent contributions and their associated finite parts, and to pinpoint the mathematical reason for the equivalence with the traditional method of regularization.

1. Introduction

Calculations of radiative corrections in quantum electrodynamics are plagued by the well known infrared divergence problem. Conventionally, one overcomes this difficulty by regularizing the infrared divergences at the S-matrix level by giving the photon an infinitesimal mass λ_{min} [1, 2]. Such divergences arise in both virtual photonic corrections and real bremsstrahlung processes. Fortunately, when the contributions from these two sources are summed to give the total corrections to transition rates, the infrared divergences cancel to any given order in perturbation theory, and the regularization parameter may be set equal to zero. This important result has been known, of course, for a long time [2]. A clear and relatively recent discussion of this cancellation can be found in refs. [1] and [3]. The authors of these papers showed that the infrared divergences could be separated from the rest of the perturbation expansion as an exponential factor. They inferred from the simple form of their exponential expression that the cancellation of lowest order infrared divergences guarantees the cancellation to all orders and that the equivalence of any two infrared regularization schemes in lowest order implies equivalence to all orders.

In this paper we make no attempt to reproduce the infrared theory [1–3] nor explore the technique of coherent states which leads to an infrared finite S-matrix [4].

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Our main interest is to examine the use of dimensional regularization in lowest order radiative corrections as a substitute scheme for λ_{min}. Such a possibility was mentioned by the originators of dimensional regularization [5, 6] and later demonstrated in some detail for the case of electron scattering by Gastmans and Meuldermans [7]. The thrust of our work, on the other hand, is two-fold: to show explicitly how dimensional regularization can lead in some cases of considerable interest to very simple and elegant evaluations of infrared divergent contributions and their associated finite parts and to pinpoint the mathematical reason for the equivalence with the λ_{min} method.

Sect. 2 of this paper outlines how dimensional regularization may be used to control infrared divergences which arise in virtual calculations at the one loop level. The application of this method is demonstrated by a simple example, the field renormalization of an electron in quantum electrodynamics. In sect. 3 we discuss the application of dimensional regularization for real bremsstrahlung calculations and show the simplicity of its use in two representative decay processes, namely the corrections to muon and π_{e2} decays associated with soft real photons. In sect. 4 we demonstrate the reason for the equivalence of the dimensional and λ_{min} techniques. Since radiative corrections in pure quantum electrodynamics and muon decay using the λ_{min} prescription have been well verified by experiment, we naturally view such equivalence as a necessary criterion for any valid regularization scheme.

2. Virtual corrections

Although the method of dimensional regularization was originally proposed as a powerful gauge invariant prescription for controlling ultraviolet divergences [5, 6], it can at the same time regularize the infrared divergences that arise in virtual photonic graphs. That is, rather than giving the photon a small mass λ_{min}, the dimension of space-time, n, can serve as a regularization parameter for both ultraviolet and infrared divergences [7]. At the one loop level both appear as simple poles at $n = 4$. In performing the one loop momentum integration, the ultraviolet divergences emerge immediately; but the infrared divergences appear only after the Feynman parameter integrations are performed. The ultraviolet poles are removed by renormalizing the parameters of the theory; but the infrared poles remain until cancelled at the transition rate level by soft bremsstrahlung contributions. Throughout this paper, we will only concern ourselves with the application of the dimensional prescription to infrared divergences.

In the original dimensional regularization prescription, all internal momentum variables of integration have n rather than 4 components; while the external momenta p_i are left as four vectors. At the one loop level, Feynman integrals take the form

$$\int \frac{d^n K}{(2\pi)^n} F(p_i, K) . \tag{2.1}$$

Various rules for manipulating this dimensionally generalized integrand and carrying out the integration are given in [5, 6, 8]. Because this integral is completely regulated by the dimension of space-time, n, we may use the massless photon propagator

$$iD_{\mu\nu} = \frac{-ig_{\mu\nu}}{K^2 + i\epsilon}\,, \tag{2.2}$$

where

$$K^2 = K_0^2 - K_1^2 - K_2^2 \ldots - K_{n-1}^2\,, \tag{2.3}$$

whenever a virtual photon line appears. Any infrared divergence originates from a Feynman parameter integral and takes the form

$$\int_0^1 \mathrm{d}x\, x^{n-5} = \frac{1}{n-4}\,, \qquad \text{for} \quad n > 4\,, \tag{2.4}$$

which when analytically continued to all n develops a simple pole at $n = 4$. Overall, this method yields the same results as the $\lambda_{\min}$ approach with the replacement

$$\ln \lambda_{\min} \leftrightarrow -\left(\frac{1}{n-4} + c\right), \tag{2.5}$$

where the constant c accompanies the pole whenever it appears and is later cancelled along with the pole. The actual value of c depends on the details of our generalization to n dimensions and may be changed

$$c \to c + \ln c_1 \tag{2.6}$$

merely by multiplying all dimensionally generalized integrals by c_1^{n-4}. Therefore, c may be taken to be zero if we wish. (See eq. (2.14) for our value of c.)

As a simple example of this technique, consider the electron self-energy amplitude

$$\Sigma(p) = -ie^2 \int \frac{\mathrm{d}^n K}{(2\pi)^n}\, \gamma^\mu \frac{1}{\not p - \not K - m_e}\, \gamma_\mu \frac{1}{K^2}\,. \tag{2.7}$$

Manipulating the integrand in n dimensional space-time, this becomes

$$\Sigma(p) = -ie^2 \int \frac{\mathrm{d}^n K}{(2\pi)^n} \frac{(2-n)(\not p - \not K) + n m_e}{(K^2 - 2p\cdot K + p^2 - m_e^2)K^2}\,, \tag{2.8}$$

or substituting $Q = K - px$

$$\Sigma(p) = -ie^2 \int_0^1 \mathrm{d}x \frac{\mathrm{d}^n Q}{(2\pi)^n} \frac{(2-n)\not p(1-x) + n m_e}{(Q^2 - C)^2}\,, \tag{2.9}$$

where $C = p^2(x^2 - x) + m_e^2 x$.

Expanding about $\not p = m_e$,

$$\Sigma(p) = A + B(\not p - m_e) + \Sigma_f(p)(\not p - m_e)^2\,. \tag{2.10}$$

The constant $B = 1 - (1/Z_2)$ (Z_2 is the field renormalization), is both ultraviolet and infrared divergent. We find from eq. (2.9)

$$B = -ie^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^n Q}{(2\pi)^n} \frac{(2-n)(1-x)}{(Q^2 - m_e^2 x^2)^2}$$
$$+ ie^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^n Q}{(2\pi)^n} \frac{4m_e^2 x(1-x)}{(Q^2 - m_e^2 x^2)^3} [(2-n)(1-x) + n]\,. \tag{2.11}$$

We recall [5, 6] that n dimensional integrals independent of direction are defined by performing the angular integrations in Q space, carrying out the radial integration in its region of convergence in n-space and analytically continuing the results to the complex n plane. With this definition we find

$$B = \frac{e^2}{(16\pi^2)^{\frac{1}{4}n}} m_e^{n-4}\, \Gamma(2 - \tfrac{1}{2}n) \left\{ \int_0^1 (2-n) x^{n-4}(1-x)\mathrm{d}x \right.$$
$$\left. + (4-n) \int_0^1 x^{n-5}(1-x)[(2-n)(1-x) + n]\,\mathrm{d}x \right\}. \tag{2.12}$$

The first and second integrals in eq. (2.12) are convergent for $n > 3$ and $n > 4$, respectively. Again we define their value for arbitrary n by carrying out the integrals in their region of convergence in n-space and analytically continuing to the complex n plane. With this definition it is immediately seen that the second term in eq. (2.12) involves a pole at $n = 4$ which manifests itself only after the parametric integration is performed. That this pole is connected with the infrared divergence can be readily ascertained by observing that the second integral in eq. (2.11), from which this contribution derives, is obviously ultraviolet convergent at $n = 4$. Performing the x integration and expanding in a Laurent series about $n = 4$ we find:

$$B = \frac{e^2}{16\pi^2} \left\{ \left(\frac{2}{n-4} + \gamma - \ln(4\pi) + 2\ln(m_e) - 4 \right) \right.$$
$$\left. + \left(\frac{4}{n-4} + 2\gamma - 2\ln(4\pi) + 4\ln(m_e) \right) + \mathrm{O}(n-4) \right\}, \tag{2.13}$$

where γ is Euler's constant. The $n = 4$ poles in the first and second parentheses correspond to the ultraviolet and infrared divergences, respectively. This result is identical to the old calculations involving $\lambda_{\min}$ if the replacement of eq. (2.5) is made in the second parenthesis of eq. (2.13) and *

$$c = \tfrac{1}{2}\gamma - \ln(2\sqrt{\pi})\,. \tag{2.14}$$

* On the other hand, the connection of the $n = 4$ pole in the first parenthesis of eq. (2.13) and the ultraviolet cutoff Λ introduced in the usual way, i.e. $1/k^2 \to 1/k^2 - 1/(k^2 - \Lambda^2)$ is,
$\ln \Lambda \leftrightarrow -\frac{1}{n-4} - \frac{1}{2}\gamma + \ln(2\sqrt{\pi}) - \frac{1}{4})$.

The simplicity of this calculation is characteristic of the applications of dimensional regularization. Radiative corrections of order α associated with proper vertex graphs are treated in exactly the same straightforward manner.

In closing this section we point out that one loop integrals that are both ultraviolet and infrared divergent such as B naturally divide into sums of integrals whose regions of convergence in n space are disjoint. The reason for this is easy to understand: integrals which are logarithmically divergent in the ultraviolet (infrared) but covergent in the infrared (ultraviolet) have a region of convergence $n < 4$ ($n > 4$). Thus, for example, examination of eqs. (2.11) and (2.12) shows that the regions of convergence of the first and second integrals in eq. (2.12) are $3 < n < 4$ and $4 < n < 6$, respectively. Thus, although the complete integral does not converge for any value of n, the integrand naturally divides into ultraviolet divergent, convergent and infrared divergent parts. Then each part may be treated separately and for each, there is some domain of n in which it exists and from which it may be continued analytically in the complex n plane.

3. Real photons

In this section we illustrate the application of the n dimensional method to the study of the soft photon contributions associated with two representative decay processes, $\mu \to e + \nu_\mu + \bar{\nu}_e + \gamma$ and $\pi \to \ell + \bar{\nu}_\ell + \gamma$ ($\ell = e$ or μ). As we will see, energy and momentum conservation impose different constraints on the energy-angle dependence of the soft photons associated with these two processes.

The prescription for the dimensional regularization of the bremsstrahlung integrations has been already spelled out by Gastmans and Meuldermans [7] and is very straightforward:

$$\frac{d^3k}{2k_0(2\pi)^3} \to \frac{d^{n-1}K}{2|K|(2\pi)^{n-1}}, \tag{3.1}$$

where $K = (K^0, K)$ is an n dimensional light-like vector:

$$K^0 = |K| = (K_1^2 + K_2^2 + ... + K_{n-1}^2)^{\frac{1}{2}}. \tag{3.2}$$

Before discussing these processes in detail we make some elementary remarks which suggest two possible strategies in carrying out the generalized integration. In the rest frame of the decaying particle, the integrand of the soft photon contributions to these decays involves the scalar product $p_2 \cdot K$ where p_2 is the four-momentum of the charged particle in the final state. Because p_2 is a four vector: $p_2 \cdot K = p_2 \cdot k$, where k is the four-vector involving only the first four components of K. Therefore, we have a choice of two different approaches in introducing polar coordinates, either

$$p_2 \cdot K = |K|(E_2 - |p_2| \cos\theta) \tag{3.3}$$

or

$$p_2 \cdot K = p_2 \cdot k = |K|E_2 - |\boldsymbol{p}_2||\boldsymbol{k}| \cos\theta', \tag{3.4}$$

where $|\boldsymbol{k}| = (K_1^2 + K_2^2 + K_3^2)^{\frac{1}{2}}$. Clearly θ is the angle between $\boldsymbol{p}_2$ and K, while θ' is the angle between $\boldsymbol{p}_2$ and the projection $\boldsymbol{k}$ of K in the subspace corresponding to the three physical space dimensions. Both procedures are of course equivalent but we have found that the first, eq. (3.3), introduces considerable simplifications. The second method based on eq. (3.4) yields integrations similar to the λ_{min} technique (see sect. 4) and is therefore of no great computational advantage but it is useful in understanding the equivalence with the traditional calculations.

3.1. Muon decay

In studying the radiative corrections of order α to muon decay it is necessary to include the inner bremsstrahlung contribution $\mu \to e + \bar{\nu}_e + \nu_\mu + \gamma$. For our purposes it will be sufficient to consider only the soft photon contributions.

In computing the decay rate, the delta function of energy-momentum conservation is used to integrate over the phase space of the unobserved neutrinos.

The infrared divergent integral becomes then:

$$I = -\int_R \frac{d^3k}{2k_0(2\pi)^3}\left(\frac{p_{2\mu}}{p_2 \cdot k} - \frac{p_{1\mu}}{p_1 \cdot k}\right)\left(\frac{p_2^\mu}{p_2 \cdot k} - \frac{p_1^\mu}{p_1 \cdot k}\right). \tag{3.5}$$

The region of integration R is determined by energy-momentum conservation, which implies that the maximum photon energy is a function of $\cos\theta$:

$$(k_0)_{max} = \frac{m_\mu(E_m - E_2)}{m_\mu - E_2 + |\boldsymbol{p}_2| \cos\theta} \geqslant E_m - E_2, \tag{3.6}$$

where $E_m = (m_\mu^2 + m_e^2)/(2m_\mu)$ is the end point energy of the electron. Note that except for the point $E_2 = E_m$, it is always possible to divide the region of integration into two intervals $0 \leqslant |\boldsymbol{k}| \leqslant \epsilon$ and $\epsilon \leqslant |\boldsymbol{k}| \leqslant |\boldsymbol{k}|_{max}$, with ϵ isotropic, i.e., independent of $\cos\theta$. In discussing the infrared divergence it is sufficient to consider the first region. We will refer to the isotropic soft photon integral as $I^{(\epsilon)}$. In the traditional treatment the divergent integral $I^{(\epsilon)}$ is given a meaning by assigning to the photon an infinitesimal mass: $k_0 = (\boldsymbol{k}^2 + \lambda_{min}^2)^{\frac{1}{2}}$. In the n dimensional approach the regularized integral is defined as

$$I_n^{(\epsilon)} = -\int_\epsilon \frac{d^{(n-1)}K}{(2\pi)^{n-1}\, 2|K|}\left(\frac{p_{2\mu}}{p_2 \cdot K} - \frac{p_{1\mu}}{p_1 \cdot K}\right)\left(\frac{p_2^\mu}{p_2 \cdot K} - \frac{p_1^\mu}{p_1 \cdot K}\right), \tag{3.7}$$

where n is the regularization parameter and the symbol ϵ reminds us that $|K|$ is constrained to lie in the ϵ region: $0 \leqslant K_0 = |K| \leqslant \epsilon$. In the muon rest frame eq. (3.7) reduces to

$$I_n^{(\epsilon)} = \int \frac{d^{n-1}K}{(2\pi)^{n-1}\, 2|K|^3} \frac{\beta^2 \sin^2\theta}{(1 - \beta\cos\theta)^2}, \tag{3.8}$$

where $\beta = |\boldsymbol{p}_2|/E_2$ and we have used eq. (3.3). Choosing the first axis along the direction of $\boldsymbol{p}_2$, θ is identified with the polar angle θ_1 (see appendix). We can then integrate over the remaining angles and obtain:

$$I_n^{(\epsilon)} = \frac{2}{(16\pi^2)^{\frac{1}{4}n}} \frac{\beta^2}{\Gamma(\frac{1}{2}n-1)} \int_0^{\epsilon} \mathrm{d}|K|\,|K|^{n-5} \int_{-1}^{+1} \mathrm{d}x \frac{(1-x^2)^{\frac{1}{2}n-1}}{(1-\beta x)^2} . \tag{3.9}$$

Performing the $|K|$ integration in the region of convergence in n space and analytically continuing the result we see that the infrared divergence has factored out and manifests itself as a pole at $n = 4$:

$$\int_0^{\epsilon} \mathrm{d}|K|\,|K|^{n-5} = \frac{\epsilon^{n-4}}{n-4} . \tag{3.10}$$

Inserting eq. (3.10) into eq. (3.9) and performing a Laurent expansion about $n = 4$:

$$I_n^{(\epsilon)} = \frac{1}{8\pi^2}\beta^2 \left[\frac{1}{n-4} + \ln(\epsilon) + \tfrac{1}{2}\gamma - \ln(2\sqrt{\pi})\right] \int_{-1}^{+1} \mathrm{d}x \frac{(1-x^2)}{(1-\beta x)^2}$$

$$+ \frac{1}{16\pi^2}\beta^2 \int_{-1}^{+1} \mathrm{d}x \frac{(1-x^2)}{(1-\beta x)^2} \ln(1-x^2) + O(n-4) . \tag{3.11}$$

Finally evaluation of the integrals leads to

$$I_n^{(\epsilon)} = \frac{1}{4\pi^2}\left\{2\left[\frac{1}{n-4} + \ln(\epsilon) + \tfrac{1}{2}\gamma - \ln(2\sqrt{\pi})\right]\left[\frac{1}{\beta}\tanh^{-1}\beta - 1\right]\right.$$

$$\left. + C(\beta) + O(n-4)\right\} , \tag{3.12a}$$

$$C(\beta) = 1 + \frac{\tanh^{-1}\beta}{\beta}\left[1 - \tanh^{-1}\beta\right] + 2\ln 2\left(\frac{1}{\beta}\tanh^{-1}\beta - 1\right)$$

$$+ \frac{1}{\beta} L\left(\frac{2\beta}{1+\beta}\right) , \tag{3.12b}$$

where $L(x) = \int_0^x (\mathrm{d}t/t)\ln|1-t|$ is the Spence function.

Detailed comparison shows that eq. (3.12) is identical to the conventional $\lambda_{\min}$ calculation * [9] provided that the identification of eqs. (2.5, 2.14) is made. Thus with our definitions the correspondence between $1/(n-4)$ and $\ln(\lambda_{\min})$ given in eqs. (2.5, 2.14) is the same for the virtual and real contributions.

It is amusing to note that the finite function $C(\beta)$ accompanying the infrared divergent term arises from the second integral in eq. (3.11) and, therefore, has its mathematical origin in the n dependence of the polar angle integrand in eq. (3.9)!

* To verify the equality of eq. (3.12b) with eq. (C.4) of ref. [9] it is convenient to use the identity: $2L(\beta) - 2L(-\beta) + L((1-\beta)/2) - L((1+\beta)/2) = 2L(2\beta/(1+\beta)) - \ln((1+\beta)/2)\ln((1+\beta)/(1-\beta))$.

3.2. $\pi_{\ell 2}$ decay

In studying the radiative corrections of order α to the decay $\pi \to \ell + \bar{\nu}_\ell$ (ℓ = e or μ) it is necessary to consider the contribution of the real photon process $\pi \to \ell + \bar{\nu}_\ell + \gamma$. We are interested in the situation in which the $\bar{\nu}_\ell$ and γ are undetected and the charged lepton is observed in the energy interval $E_m - \Delta E \leqslant E \leqslant E_m$ where $E_m = (m_\pi^2 + m_\ell^2)/(2m_\pi)$ is the energy of the lepton in the two-body decay and $\Delta E \ll E_m$. We will retain terms of logarithmic and zeroth order in ΔE but will neglect terms of order ΔE or higher. As only soft photons contribute in this limit, the effects of the strong interactions in the contributions from $\pi \to \ell + \bar{\nu}_\ell + \gamma$ can be neglected.

With the usual $\lambda_{\min}$ regularization method the differential decay rate in the π rest frame is given by:

$$\mathrm{d}P = \frac{G_{\mathrm{V}}^2}{2\pi^3}\frac{1}{m_\pi} e^2 f_\pi^2 m_\ell^2 \frac{\mathrm{d}^3p}{2E} p^2 (m_\pi E - m_\ell^2)$$

$$\times \int \frac{\mathrm{d}^3k}{2k_0(2\pi)^3} \frac{(1 - (|\boldsymbol{k}|^2/k_0^2)\cos^2\theta)}{(Ek_0 - p|\boldsymbol{k}|\cos\theta)^2} \delta(G^2) , \tag{3.13}$$

where $k_0^2 = |\boldsymbol{k}|^2 + \lambda_{\min}^2$, $G_{\mathrm{V}} = G_\mu \cos\theta_{\mathrm{c}}$ and f_π are the weak vector and pion decay coupling constants, $\ell = (E, \boldsymbol{p})$ is the charged lepton four-momentum and $G = p_\pi - \ell - k$. In eq. (3.13) we have integrated over the unobserved neutrino phase space, summed over polarizations and discarded terms of order ΔE and higher.

In the dimensional regularization method we set $\lambda_{\min} = 0$ from the beginning but generalize the integral to $n-1$ dimensions according to eq. (3.1). Thus

$$\mathrm{d}P = \frac{G_{\mathrm{V}}^2}{2\pi^2}\frac{1}{m_\pi} e^2 f_\pi^2 m_\ell^2 \frac{\mathrm{d}^3p}{2E} (m_\pi E - m_\ell^2)\beta^2$$

$$\times \int \frac{\mathrm{d}^{n-1}K}{2|K|^3(2\pi)^{n-1}} \frac{(1-\cos^2\theta)}{(1-\beta\cos\theta)^2} \delta(G^2) , \tag{3.14}$$

where we have used eq. (3.3) and $\beta = p/E$. Choosing again the first axis along the direction of $\boldsymbol{p}$ the integrand depends only on the angle $\theta_1 = \theta$ so that we can immediately perform the other angular integrations. Further using the $\delta(G^2)$ to carry out the radial $\mathrm{d}|K|$ integration, we obtain for the probability of emitting ℓ in the interval $E_m - \Delta E \leqslant E \leqslant E_m$:

$$\Delta P(E > E_m - \Delta E) = \frac{G_{\mathrm{V}}^2 e^2 f_\pi^2 m_\ell^2}{\pi(16\pi^2)^{\frac{1}{4}n}\Gamma(\frac{1}{2}n-1)} \int_{E_m-\Delta E}^{E_m} \mathrm{d}E (E_m - E)^{n-5} p\beta^2 \frac{m_\pi E - m_\ell^2}{m_\pi^2}$$

$$\times \int_{-1}^{+1} \mathrm{d}x \frac{(1-x^2)^{\frac{1}{2}n-1}}{(1-\beta x)^2} \left(1 - \frac{E}{m_\pi}(1-\beta x)\right)^{4-n} . \tag{3.15}$$

Expanding everything to the right of $(E_m - E)^{n-5}$ in a Taylor series about $E = E_{max}$ and changing variables to $u = E_m - E$:

$$\Delta P = \frac{G_V^2 e^2 f_\pi^2 m_\ell^2}{\pi(16\pi^2)^{\frac{1}{4}n}\,\Gamma(\frac{1}{2}n-1)\,m_\pi^2} \int_0^{\Delta E} du\, u^{n-5}\{A + Bu + Cu^2 \ldots\}\,. \qquad (3.16)$$

It is easy to see that in the limit $n \to 4$ the terms $Bu + Cu^2 + \ldots$ contribute finite integrals of order ΔE and higher. Therefore we neglect them and obtain

$$\Delta P = \frac{G_V^2 e^2 f_\pi^2 m_\ell^2}{\pi(16\pi^2)^{\frac{1}{4}n}\,\Gamma(\frac{1}{2}n-1)\,m_\pi^2}\,\frac{(\Delta E)^{n-4}}{n-4}\,A\,, \qquad (3.17a)$$

$$A = \left(\frac{1-\mu^2}{2}\right)^{6-n} m_\pi^3 \beta_m^2 \int_{-1}^{+1} dx\, \frac{(1-x^2)}{(1-\beta_m x)^2}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}n-1}\,, \qquad (3.17b)$$

where $\mu = (m_\ell/m_{\pi^+})$ and $\beta_m = (1-\mu^2)/(1+\mu^2)$. Expanding eq. (3.17) in a Laurent series about $n = 4$ and performing the ensuing elementary integrals we finally obtain

$$\Delta P = \frac{2\alpha}{\pi} P_0\left(\frac{1}{n-4} + \ln \Delta E + \tfrac{1}{2}\gamma - \ln(2\sqrt{\pi}) - \ln\left(\frac{1-\mu^2}{2}\right) + \tfrac{1}{2}\ln\mu\right)$$

$$\times\left(\frac{1+\mu^2}{1-\mu^2}\ln\left(\frac{1}{\mu}\right) - 1\right) + O(n-4)\,, \qquad (3.18)$$

where

$$P_0 = \frac{G_V^2}{8\pi} f_\pi^2 m_\ell^2 m_{\pi^+}(1-\mu^2)^2 \qquad (3.19)$$

is the uncorrected decay rate for $\pi \to \ell + \bar{\nu}_\ell$.

Again with the identification of eqs. (2.5, 2.14) we recover the results of the traditional λ_{min} calculations [10], which are of considerable interest for the verification of e – μ universality. The same basic calculation holds for the decay of the intermediate W boson [11] $W \to \ell + \bar{\nu}_e + \gamma$, where a correct calculation of the finite parts of eq. (3.18) is important to verify the theorem on mass singularities [9, 12].

We have illustrated these two examples in considerable detail to show the simplicity and elegance of the n dimensional regularization of infrared divergences. However, it is not particularly transparent why the traditional and the new regularization schemes give the same finite answers. We address ourselves to this question in sect. 4.

4. Equivalence of the dimensional and λ_{min} schemes

In this section we demonstrate the reason for the equivalence of the λ_{min} and dimensional regularization schemes by exploiting a strong similarity between these two approaches.

In the λ_{min} prescription k^2 is always replaced by

$$k^2 \to k^2 - \lambda_{min}^2\,, \quad (k^2 = k_0^2 - k_1^2 - k_2^2 - k_3^2)\,, \qquad (4.1)$$

while in the dimensional generalization it is replaced by

$$k^2 \to K^2 = k^2 - \omega^2\,, \quad (\omega^2 = K_4^2 + K_5^2 + \ldots + K_{n-1}^2)\,. \qquad (4.2)$$

This points out the close connection between the photon mass λ_{min} and the magnitude of the extra components of K. In fact, the introduction of the extra components, the integration over these variables and the further limit $n \to 4$ look very much like giving the photon a mass ω and then in some sense applying to such expression an operator that picks up only the very low ω dependence of the finite parts. We illustrate this in the following examples.

4.1. Virtual graphs

We return to the example of sect. 2 and consider the infrared divergent part of eq. (2.11). It can be written as:

$$B_{inf} = \int \frac{d^{n-4}\omega}{(2\pi)^{n-4}}\, ie^2 \int_0^1 dx \int \frac{d^4q}{(2\pi)^4}\,\frac{8m_e^2 x}{(q^2-\omega^2-m_e^2x^2)^3}\,, \qquad (4.3)$$

where we have separated the integration over the additional $n-4$ spatial variables and replaced $Q^2 = q^2 - \omega^2$. Performing the d^4q and angular integrations of $d^{n-4}\omega$ we have

$$B_{inf} = \frac{2(16\pi^2)^{1-\frac{1}{4}n}}{\Gamma(\frac{1}{2}n-2)}\int_0^\infty d\omega\,\omega^{n-5}\left\{\frac{2e^2}{16\pi^2}\int_0^1 dx\,\frac{2m_e^2 x}{m_e^2x^2+\omega^2}\right\}. \qquad (4.4)$$

The bracketed expression in eq. (4.4) would produce exactly the usual λ_{min} result if we replaced $\omega \to \lambda_{min}$. To analyze the effect of the operator

$$[2(16\pi^2)^{1-\frac{1}{4}n}/\Gamma(\tfrac{1}{2}n-2)]\int_0^\infty d\omega\,\omega^{n-5}\,,$$

we note that the factor $1/\Gamma(\frac{1}{2}n-2)$ contains a zero at $n = 4$, and that therefore only terms containing poles at $n = 4$ after the ω integration is performed can contribute. This suggests dividing the integration over ω into two intervals which may be conveniently taken to be $0 \leqslant \omega \leqslant m_e$ and $m_e \leqslant \omega \leqslant \infty$. The ω integration in the second interval is perfectly finite as $n \to 4$ and thus its contribution vanishes at $n = 4$ when multiplied by $1/\Gamma(\frac{1}{2}n-2)$. To study the effect of the first region of or in terms of $x = \cos\theta$

$$\frac{2\pi^{\frac{1}{2}n-1}}{\Gamma(\frac{1}{2}n-1)} \int \mathrm{d}|K| \int_{-1}^{1} \mathrm{d}x\, |K|^{n-2}(1-x^2)^{\frac{1}{2}n-2} \,. \tag{A.6}$$

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ON THE HIGH-ENERGY BEHAVIOUR OF BORN CROSS SECTIONS IN QUANTUM GRAVITY

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The differential cross sections for elastic graviton-scalar, graviton-photon, and graviton-graviton scattering are calculated in lowest order. It is found that they have a maximal growth with energy allowed on dimensional grounds, i.e., like E^2. It is argued that this behavior does not exclude the renormalizability of the theory.

1. Introduction

Quantum gravity has a peculiar place in elementary particle physics. On the one hand, it is important to investigate the theory as it describes how gravity couples to the elementary particles, and this has to do with some of the most fundamental aspects of nature. On the other hand, the almost incredible smallness of the coupling constant makes a detailed study of the effects due to gravity on elementary particle processes rather academic. These two opposing aspects of the theory make most physicists quite hesitant to examine it more deeply.

Our point of view has been that as long as no satisfactory, renormalizable theory has been formulated describing the interaction of the quantized gravitational field and the matter fields, one is left with the embarassing question of what one is really missing or overlooking. Exactly this missing piece of knowledge could prove to be very valuable for the understanding of the other, experimentally more accessible fundamental interactions. This is really the justification of the work we present here.

More specifically, we wanted to get a hint about which set of elementary particles could be coupled in a meaningful way to the graviton. In this context, we have already shown that quantum gravity corrections to the anomalous magnetic moment of the electron and the muon are finite at the one-loop level [1]. This time, how-

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