RGE of Wilson coefficients and operators (simplified to no-mixing) (1

it is important to be consistent with the definition of renormalization constant.
Expressions that look different may be equivalent if the renormalization constants are defined differently.

Let's consider the generic case of a series of effective operators added to a renormalizable $\mathcal{L}$:

$$\mathcal{L} \to \mathcal{L} + \sum_i c_i O_i$$

$\downarrow$ going through the renormalization procedure, we can treat any $c_i O_i$ as a new interaction and renormalize it by introducing a corresponding Z_O (after renormalizing masses, couplings, and fields in $\mathcal{L}$, of which O_i are also functions).

$c_i O_i \to$ interpret them first as $c_i^B O_i^B$ ($B \to$ "bare")

$\downarrow$ in terms of renormalized couplings, masses and fields

$\mathcal{L} \to \mathcal{L}_{QED}$

assume NO-MIXING for simplicity

$$c_i^B O_i^B = c_i(e,m) \underbrace{Z_{O_i} O_i(\psi, A)}_{O_i^R(\psi, A, m, \ldots)}$$

if we define:

$$c_i^B = Z_i c_i(e,m)$$

$$O_i^B = Z_\psi^{n_\psi/2} Z_A^{n_A/2} O_i(\psi, A) = \boxed{\frac{Z_\psi^{n_\psi/2} Z_A^{n_A/2}}{Z_{O_i}}} O_i^R(\psi, A, \ldots)$$

$$\equiv Z_i^{OP} O_i^R(\psi, A, \ldots)$$

$$C_1^B O_1^B = C_i^R \cdot O_i^R \longrightarrow Z_i = (Z_i^{op})^{-1}$$

$$\hookrightarrow Z_i C_i^R \cdot Z_i^{op} \cdot O_i^R$$

so, depending if one defines γ_i in terms of Z_i or Z_i^{op}, the signs of the corresponding RGE will be different (but the signs of Z_i and Z_i^{op} will also be! so everything should turn out to be the same in terms of scale-dependence).

$\hookrightarrow$ {notice that we are particularly interested into $C_i(\mu)$.}

The corresponding RGE are obtained as follows:

(1) $C_i^B = Z_i C_i^R \longrightarrow \mu \dfrac{dC_i^B}{d\mu} = 0 \longrightarrow \dfrac{1}{C_1^R} \mu \dfrac{dC_1^R}{d\mu} = -\dfrac{1}{Z_i} \mu \dfrac{dZ_i}{d\mu} =$ $\;Z_i = (Z_i^{op})^{-1}$

$$= \frac{1}{Z_1^{op}} \mu \frac{dZ_i^{op}}{d\mu} \equiv \gamma_i^{op}$$

$$\boxed{\mu \frac{dC_1^R(\mu)}{d\mu} = \gamma_i^{op} C_1^R(\mu)}$$

(2) $O_1^B = Z_1^{op} O_1^R \longrightarrow \mu \dfrac{dO_1^B}{d\mu} = 0 \longrightarrow \dfrac{1}{O_1^R} \mu \dfrac{dO_1^R}{d\mu} = -\dfrac{1}{Z_1^{op}} \mu \dfrac{dZ_i^{op}}{d\mu} = -\gamma_1^{op}$

$$\boxed{\mu \frac{dO_1^R(\mu)}{d\mu} = -\gamma_i^{op} O_1^R(\mu)}$$

You can see this at work directly in the case of the mass operator, where:

$$C_m = m$$
$$O_m = \bar{\psi}\psi$$

$$m_0 \bar{\psi}_0 \psi_0 = C_m^B O_m^B = m Z_m \cdot Z_\psi \bar{\psi}\psi = m Z'_m \underbrace{\bar{\psi}\psi}_{O_m^R}$$

$$\begin{cases} m_0 = Z_m \cdot m \\ O_m^B = \bar{\psi}_0 \psi_0 = Z_\psi \bar{\psi}\psi = \dfrac{Z_\psi}{Z'_m} O_m^R = \dfrac{1}{Z_m} O_m^R \end{cases}$$

$$\hookrightarrow Z_m^{op} = Z_m^{-1}$$

① $$\mu \frac{d m_0}{d\mu} = 0 \rightarrow \frac{1}{m} \mu \frac{dm}{d\mu} = -\frac{1}{Z_m} \mu \frac{d Z_m}{d\mu} = \frac{1}{Z_m^{op}} \mu \frac{d Z_m^{op}}{d\mu} = \gamma_m$$

② $$\mu \frac{d O_m^B}{d\mu} = 0 \longrightarrow \frac{1}{O_m^R} \mu \frac{d O_m^R}{d\mu} = -\frac{1}{Z_m^{op}} \mu \frac{d Z_m^{op}}{d\mu} = -\gamma_m$$

And the same also applies to the QED interaction op.: $e\bar{\psi}\gamma^\mu A_\mu \psi$

$$C_e = e \qquad\qquad C_e^B = e_0 = Z_e \cdot e$$

$$O_e = \bar{\psi}\gamma^\mu A_\mu \psi \qquad\qquad O_e^B = \bar{\psi}_0 \gamma^\mu A_{\mu 0} \psi_0 = \frac{Z_\psi Z_A^{1/2}}{Z_1} O_{e,R} = \frac{1}{Z_e} O_{e,R}$$

and we can think of the RGE for e and O_e as follows:

$$\mu \frac{d e_0}{d\mu} = 0 \rightarrow \frac{1}{e} \mu \frac{de}{d\mu} = -\frac{1}{Z_e} \mu \frac{d Z_e}{d\mu} = \beta(e) \cdot \frac{1}{e} = \text{"}\gamma_e\text{"}$$

etc.