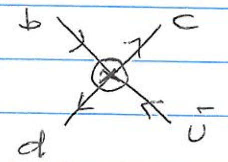
Lesson 4 (continuing from lesson 3)

We now have to repeat the same calculation using $\mathcal{L}_{EFT}$ and then impose that $\mathcal{A}_{EFT} = \mathcal{A}_{SM}$ at a given matching scale $\tilde{\mu}$.

$\hookrightarrow$ we have the analogous six diagrams (in which the W has been removed), in which we use F.R. dictated by $\mathcal{L}_{EFT}$, and one CT diagram corresponding to:

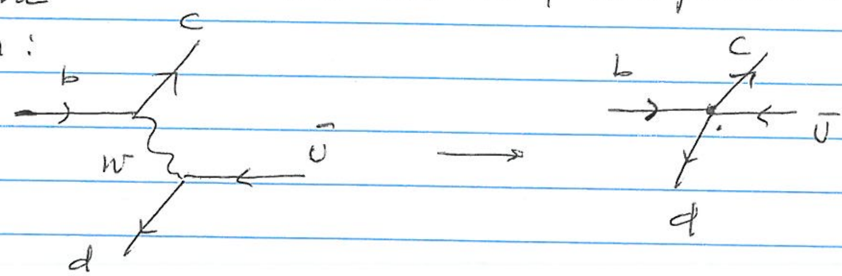


{which now includes both O_1 and O_2!}

$\swarrow$

we can determine it directly from the calculation of the UV-divergent part of the six one-loop diagrams of the EFT:

using the notation:



etc.

(a) in SM $\longrightarrow$ (a') in EFT

we find that:

$$\mathcal{A}_{(a')+\dots+(f')} = \frac{\alpha_s}{2\pi}\left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}}\right)\left(3C_2 - \frac{11}{3}C_1\right)(\bar{c}_L\gamma^\mu b_L)(\bar{d}_L\gamma_\mu u_L)$$

$$+ \frac{\alpha_s}{2\pi}\left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}}\right)\left(3C_1 - \frac{11}{3}C_2\right)(\bar{c}_L\gamma^\mu u_L)(\bar{d}_L\gamma_\mu b_L)$$

$\swarrow$

$$\mathcal{A}_{CT} = -\frac{\alpha_s}{2\pi}\frac{1}{\epsilon_{UV}}\left[\left(3C_2 - \frac{11}{3}C_1\right)(\bar{c}_L\gamma^\mu b_L)(\bar{d}_L\gamma_\mu u_L) + \left(3C_1 - \frac{11}{3}C_2\right)(\bar{c}_L\gamma^\mu u_L)(\bar{d}_L\gamma_\mu b_L)\right]$$

from which:

$$\mathcal{A}_{EFT} = -\left[C_1 + \frac{\alpha_s}{2\pi}\frac{1}{\epsilon_{IR}}\left(3C_2 - \frac{11}{3}C_1\right)\right](\bar{c}_L\gamma^\mu b_L)(\bar{d}_L\gamma_\mu u_L) +$$
$$-\left[C_2 + \frac{\alpha_s}{2\pi}\frac{1}{\epsilon_{IR}}\left(3C_1 - \frac{11}{3}C_2\right)\right](\bar{c}_L\gamma^\mu u_L)(\bar{d}_L\gamma_\mu b_L)$$

Matching: impose that $\mathcal{A}_{SM}$ (on p.8) and $\mathcal{A}_{EFT}$ above give the same result at a given matching scale $\tilde{\mu}$.

$\downarrow$

since $\mathcal{A}_{SM}$ contains $\ln\frac{\tilde{\mu}}{M_W^2}$ it will be in general convenient to choose a scale $\tilde{\mu} \simeq M_W$.

In general, the matching procedure determines $C_1(\tilde{\mu})$ and $C_2(\tilde{\mu})$ as (see p. 10')

$$C_1(\tilde{\mu}) = G\left[1 - \frac{\alpha_s}{2\pi}\left(\frac{1}{2}\ln\frac{\tilde{\mu}^2}{M_W^2} + \frac{3}{4}\right)\right] \quad \leftarrow \text{starts at } O(1)$$

$$C_2(\tilde{\mu}) = G\left[\frac{3\alpha_s}{2\pi}\left(\frac{1}{2}\ln\frac{\tilde{\mu}^2}{M_W^2} + \frac{3}{4}\right)\right] \quad \leftarrow \text{starts at } O(\alpha_s)$$

notice, as anticipated, that $C_1(\tilde{\mu})$ and $C_2(\tilde{\mu})$ are IR finite

$\hookrightarrow$ $\mathcal{L}_{SM}$ and $\mathcal{L}_{EFT}$ describe the same IR physics!

Now comes the interesting part: C_1 and C_2 are just couplings in our $\mathcal{L}_{EFT}$, and as such they end up being scale-dependent objects through the renormalization of the EFT they belong to. We need to treat them as any other coupling!

$\hookrightarrow$ and can therefore calculate their scale evolution via renormalization + RGE.

$\boxed{\text{idea}}$: use the matching calculation to determine $C_1(M_W)$ and $C_2(M_W)$ and then evolve them by RGE to obtain $C_1(m_b)$ and $C_2(m_b)$, i.e. their value at the scale where $b \to c\bar{u}d$ lives.

$\hookrightarrow$ the RGE solution will resum large logarithmic corrections

Matching calculation:

$$-C_1\left[1-\frac{\alpha_s}{2\pi}\frac{1}{\epsilon_{IR}}\frac{11}{3}\right]-C_2\frac{\alpha_s}{2\pi}\frac{3}{\epsilon_{IR}}=-G\left[1-\frac{\alpha_s}{2\pi}\left(\frac{11}{3}\frac{1}{\epsilon_{IR}}+\frac{1}{2}\ln\frac{\tilde{\mu}^2}{M_W^2}+\frac{3}{4}\right)\right]$$

$$-C_2\left[1-\frac{\alpha_s}{2\pi}\frac{1}{\epsilon_{IR}}\frac{11}{3}\right]-C_1\frac{\alpha_s}{2\pi}\frac{3}{\epsilon_{IR}}=-G\frac{3\alpha_s}{2\pi}\left[\frac{1}{\epsilon_{IR}}+\frac{1}{2}\ln\frac{\tilde{\mu}^2}{M_W^2}+\frac{3}{4}\right]$$

Remember that $C_1=C_{1,0}+\alpha_s C_{1,1}=G+C_{1,1}\alpha_s$

$$C_2=C_{2,1}\alpha_s$$

and we need to work out the $O(\alpha_s)$ matching.

$\downarrow$

$$-(G+C_{1,1}\alpha_s)\left[1-\frac{\alpha_s}{2\pi}\frac{1}{\epsilon_{IR}}\frac{11}{3}\right]=-G\left[1-\frac{\alpha_s}{2\pi}\left(\frac{11}{3}\frac{1}{\epsilon_{IR}}+\frac{1}{2}\ln\frac{\tilde{\mu}^2}{M_W^2}+\frac{3}{4}\right)\right]$$

$$-C_{2,1}\alpha_s-(G+\cancel{C_{1,1}\alpha_s})\frac{\alpha_s}{2\pi}\frac{3}{\epsilon_{IR}}=-G\frac{3\alpha_s}{2\pi}\left[\frac{1}{\epsilon_{IR}}+\frac{1}{2}\ln\frac{\tilde{\mu}^2}{M_W^2}+\frac{3}{4}\right]$$

$\downarrow$

$$-\cancel{G}+G\frac{\alpha_s}{2\pi}\cancel{\frac{1}{\epsilon_{IR}}\frac{11}{3}}-C_{1,1}\alpha_s=-\cancel{G}+G\frac{\alpha_s}{2\pi}\left(\cancel{\frac{11}{3}\frac{1}{\epsilon_{IR}}}+\frac{1}{2}\ln\frac{\tilde{\mu}^2}{M_W^2}+\frac{3}{4}\right)$$

$$\hookrightarrow C_{1,1}\alpha_s=-G\frac{\alpha_s}{2\pi}\left(\frac{1}{2}\ln\frac{\tilde{\mu}^2}{M_W^2}+\frac{3}{4}\right)$$

$$-C_{2,1}\alpha_s-\cancel{G\frac{\alpha_s}{2\pi}\frac{3}{\epsilon_{IR}}}=-G\frac{3\alpha_s}{2\pi}\left(\cancel{\frac{1}{\epsilon_{IR}}}+\frac{1}{2}\ln\frac{\tilde{\mu}^2}{M_W^2}+\frac{3}{4}\right)$$

$$\hookrightarrow C_{2,1}\alpha_s=G\frac{3\alpha_s}{2\pi}\left(\frac{1}{2}\ln\frac{\tilde{\mu}^2}{M_W^2}+\frac{3}{4}\right)$$

Deriving the RGE equation for $c_1(\mu)$ and $c_2(\mu)$:

Consider the (O_1, O_2) portion of $\mathcal{L}_{EFT}$:

$$\mathcal{L}_{EFT} = \dots \quad - c_1 O_1 - c_2 O_2$$

and go through the usual (multiplicative) renormalization procedure, according to which you first consider the terms in the Lagrangian to be expressed in terms of "bare" fields and couplings, and you redefine them in terms of the corresponding "renormalized" objects.

$$\mathcal{L}_{EFT} = \dots - \underbrace{c_{1,0} O_{1,0}} - c_{2,0} O_{2,0}$$

let's focus on one term, the other is analogous
"0" indices indicate "bare" objects, while objects with no indices will be the renormalized ones.

Using that:

$$O_{1,0} = (\bar{c}_{0L} \gamma^\mu b_{0L})(\bar{d}_{0,L} \gamma_\mu u_{0,L}) =$$

$$= Z_\psi^2 (\bar{c}_L \gamma^\mu b_L)(\bar{d}_L \gamma_\mu u_L) = Z_\psi^2 O_1$$

field renormalization constant

$$Z_\psi = 1 - \frac{\alpha_s}{2\pi} \frac{1}{\epsilon_{UV}} \cdot \frac{4}{3} \quad \text{(from QCD)}$$

$$c_{1,0} = Z_{c_1} c_1 \quad \left(\text{analogous of } g_{s,0} = Z_{g_s} g_s \text{ in QCD}\right)$$

(or $e_0 = Z_e e$ in QED)

we rewrite:

$$-c_{1,0} O_{1,0} = -c_1 \underbrace{Z_{c_1} Z_\psi^2}_{Z_1} O_1 = -c_1 Z_1 O_1$$

$\boxed{Z_1 \text{ can be derived from p. 9 (VACT)}}$

and define

$$Z_1 = Z_{c_1} Z_\psi^2 \longrightarrow \text{analogous of } Z_{vertex} \text{ in QCD or QED}$$

Since bare couplings are scale independent:

$$\mu \frac{d}{d\mu} C_{1,0} = 0 = \mu \frac{d}{d\mu}\left(C_1 Z_{C_1}\right) = \mu \frac{d}{d\mu}\left(Z_1 Z_\varphi^{-2} C_1\right)$$

from this equation we can derive:

$$\mu \frac{dC_1}{d\mu} = \cdots$$

through some simple steps you can get:

$$\mu \frac{d}{d\mu}C_1 + C_1 \frac{1}{Z_1}\mu \frac{dZ_1}{d\mu} + C_1 \frac{1}{Z_\varphi^{-2}} \mu \frac{d}{d\mu} Z_\varphi^{-2} = 0$$

$$\mu \frac{d}{d\mu}C_1 + C_1 \mu \frac{d\ln Z_1}{d\mu} + C_1 \mu \frac{d\ln Z_\varphi^{-2}}{d\mu} = 0$$

$$\mu \frac{d}{d\mu}C_1 + C_1 \mu \frac{d\ln\left(Z_1 Z_\varphi^{-2}\right)}{d\mu} = 0$$

Now remember that: $Z_i = 1 + \delta Z_i$ (any i)

δZ_i is an expansion in α_s and we are working at the first order in α_s.

all previous expressions can be "linearized" in α_s $\rightarrow$

(on p. 11)

$$\mu \frac{dC_1}{d\mu} + C_1 \mu \frac{d\ln\left(1 + \delta Z_1 - 2\delta Z_\varphi\right)}{d\mu} = 0$$

$$\downarrow$$

$$\mu \frac{dC_1}{d\mu} + C_1 \left(\frac{\partial \delta Z_1}{\partial \alpha_s} - 2\frac{\partial \delta Z_\varphi}{\partial \alpha_s}\right) \mu \frac{d\alpha_s}{d\mu} = 0$$

QCD β-function $\leftarrow$ $\beta(\alpha_s) = -\epsilon\alpha_s - \frac{\alpha_s^2}{2\pi}\beta_0$

δZ_φ is given on p. 11.

δZ_1 can be derived from the $O(\alpha_s)$ calculation of $\mathcal{L}_{EFT}$.

↳ see $\mathcal{L}_{CT}$ on p. 9, from where:

$$Z_1 = 1 + \frac{\alpha_s}{2\pi}\frac{1}{\epsilon_{UV}}\left(3\frac{C_2}{C_1} - \frac{11}{3}\right)$$

and analogously (for future use):

$$Z_2 = 1 + \frac{\alpha_s}{2\pi}\frac{1}{\epsilon_{UV}}\left(3\frac{C_1}{C_2} - \frac{11}{3}\right)$$

Using δZ_φ and δZ_1 above we get:

$$\mu\frac{dC_1}{d\mu} = \frac{\alpha_s}{2\pi}(3C_2 - C_1)$$

and with an analogous procedure:

$$\mu\frac{dC_2}{d\mu} = \frac{\alpha_s}{2\pi}(3C_1 - C_2)$$

which can be expressed in a unique equation as:

non-diagonal = operator mixing

$$\mu\frac{dC_i}{d\mu} = \frac{\alpha_s}{2\pi}\gamma_{ij}C_j$$

Here is what we mean by "mixing under renormalization/evolution" →

↳ $\gamma = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}$ → "anomalous dimension" matrix

The obvious procedure to solve this system of equations is to diagonalize it.

↳ eigenvalues: $\begin{cases} \gamma_0 = 2 \\ \gamma_3 = -4 \end{cases}$, diagonalization matrix: $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

the labelling refers to color properties. Take it as such!

eigenvectors: $C_0 = C_1 + C_2$
$C_3 = C_1 - C_2$

the decoupled set of equations is:

$$\begin{cases} \mu \dfrac{dC_0}{d\mu} = \dfrac{\alpha_s}{2\pi} \gamma_0 C_0 \\ \\ \mu \dfrac{dC_3}{d\mu} = \dfrac{\alpha_s}{2\pi} \gamma_3 C_3 \end{cases} \xrightarrow{\text{solve for}} \{C_0, C_3\} \longrightarrow \begin{aligned} C_1 &= \frac{C_0+C_3}{2} \\ C_2 &= \frac{C_0-C_3}{2} \end{aligned}$$

Let's see what the solution looks like.
Consider the equation for C_0. We can rearrange it as follows:

$$\frac{dC_0}{C_0} = \gamma_0 \frac{\alpha_s(\mu)}{2\pi} \frac{1}{\mu} d\mu$$

$$= \gamma_0 \frac{\alpha_s/2\pi \, d\alpha_s}{\beta(\alpha_s)} = \frac{\overset{\nearrow \gamma_0 \alpha_s/2\pi}{\gamma(\alpha_s)}}{\beta(\alpha_s)} d\alpha_s$$

$$d\alpha_s = \frac{1}{\mu} \underbrace{\mu \frac{d\alpha_s}{d\mu}}_{\beta(\alpha_s)} d\mu = \beta(\alpha_s) \frac{1}{\mu} d\mu$$

$$\ln C_0 \Big|_{\mu}^{M_W} = \int_{\alpha_s(\mu)}^{\alpha_s(M_W)} \frac{\gamma(\alpha_s)}{\beta(\alpha_s)} d\alpha_s$$

expand it at the $O(\alpha_s)$ at which the calculation is being performed
in our case: $O(\alpha_s)$

$$\frac{C_0(M_W)}{C_0(\mu)} = \exp\left[\int_{\alpha_s(\mu)}^{\alpha_s(M_W)} \frac{\gamma(\alpha_s)}{\beta(\alpha_s)} d\alpha_s\right]$$

$$C_0(\mu) = C_0(M_W) \cdot \exp\left[\int_{\alpha_s(M_W)}^{\alpha_s(\mu)} \frac{\gamma(\alpha_s')}{\beta(\alpha_s')}\, d\alpha_s'\right]$$

$O(\alpha_s)$

$$\frac{\gamma(\alpha_s)}{\beta(\alpha_s)} = \frac{\gamma_0\, \alpha_s/2\pi}{-\beta_0\, \alpha_s^2/2\pi} = -\frac{\gamma_0}{\beta_0}\frac{1}{\alpha_s}$$

$\gamma_0 = 2$

$\beta_0 = \frac{23}{3}$ $(N_f = 5)$

$$C_0(\mu) = C_0(M_W) \exp\left[\int_{\alpha_s(M_W)}^{\alpha_s(\mu)} \left(-\frac{6}{23}\right) \frac{d\alpha_s'}{\alpha_s'}\right]$$

$$\exp\left[-\frac{6}{23} \ln\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)\right]$$

$$\boxed{C_0(\mu) = C_0(M_W)\left[\frac{\alpha_s(M_W)}{\alpha_s(\mu)}\right]^{6/23}}$$

and in the same way we get:

$$\boxed{C_3(\mu) = C_3(M_W)\left[\frac{\alpha_s(M_W)}{\alpha_s(\mu)}\right]^{-12/23}}$$

RGE equations for C_0, C_3 → can get C_1, C_2 at any scale. in particular: $C_1(m_b)$, $C_2(m_b)$

Notice that

$$\underbrace{\left[\frac{\alpha_s(M_W)}{\alpha_s(\mu)}\right]^{a}}_{O(\alpha_s)} = \left[1 + \frac{\alpha_s(M_W)}{2\pi}\beta_0 \ln\frac{\mu}{M_W}\right]^{a}$$

→ whose expansion will RESUM terms of the form

also known as "leading logarithms" ← $\left(\alpha_s \ln\frac{\mu}{M_W}\right)^{n}$ ← $\mu = m_b$: $\alpha_s \ln\left(\frac{m_b}{M_W}\right)$

The Moral of this example is:

↳ by using $\mathcal{L}_{EFT}$ in a case when indeed we knew the complete UV theory, we have been able to obtain a more precise perturbative prediction for low-energy interactions.
the increased precision is due to the possibility of resumming an entire family of corrections (the first member of which appear in the $O(\alpha_s)$ calculation that we have performed) thanks to the RGE formalism.

↓

since:

$$\Gamma_{th}(\bar{B}^0 \to D^+\pi^-) \propto \left(\sqrt{C_1^2+C_2^2}\right)^2_{\mu=m_b} \cdot |V_{cb}|^2 \cdots$$

↓

an increased precision in the determination of $C_1(m_b)$ and $C_2(m_b)$ will allow to extract V_{cb} from a measurement of $\Gamma(\bar{B}^0 \to D^+\pi^-)$ with increased accuracy.

↳ and check SM consistency with CKM paradigm

↓

If not → $\boxed{\text{indirect evidence of new physics beyond SM}}$

In numbers:

$C_1(m_b) \simeq 1.08G$

$C_2(m_b) \simeq -0.189G$

↳ $\sqrt{C_1^2+C_2^2}\,(m_b) \simeq 1.09G$ (would be G at tree level and $0.98G$ at 1-loop with no resummat.)

↓

9% effect

18% effect on V_{cb} ←

⟷ BIG