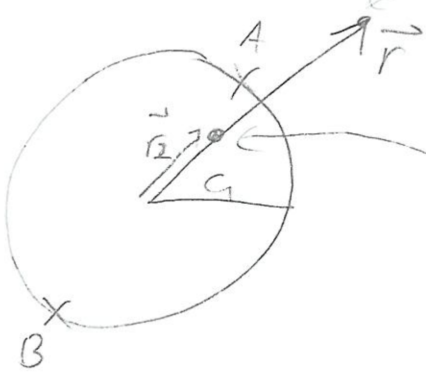
Spherical Images

Can cancel potential on sphere with a single extra "charge" on inside (for outside solutions, vice-versa for inside sol.)



original "charge" $q=+1$

image "charge" q_2
must be in same direction if cancels with only this one

Need two equations to solve for $|\vec{r}_2|$, q_2

$$V_A = 0 = \frac{-1}{4\pi|\vec{r}-r_A|} + \frac{-q_2}{4\pi|\vec{r}_2 - r_A|} \rightarrow \frac{-1}{4\pi(|\vec{r}|-a)} + \frac{-q_2}{4\pi(a-|\vec{r}_2|)} = 0$$

$$V_B = 0 = \frac{-1}{4\pi|\vec{r}-\vec{r}_B|} + \frac{-q_2}{4\pi|\vec{r}_2-\vec{r}_B|} \rightarrow \frac{-1}{4\pi(|\vec{r}|+a)} + \frac{-q_2}{4\pi(a+|\vec{r}_2|)} = 0$$

$$q_2 = \frac{-4\pi(a-|\vec{r}_2|)}{4\pi(|\vec{r}|-a)} = \frac{-4\pi(a+|\vec{r}_2|)}{4\pi(|\vec{r}|+a)}$$

$$(a-|\vec{r}_2|)(|\vec{r}|+a) = (a+|\vec{r}_2|)(|\vec{r}|-a)$$

$$|\vec{r}_2|(-|\vec{r}|-a-(|\vec{r}|-a)) = a(|\vec{r}|-a-(|\vec{r}|+a))$$

$$-2|\vec{r}||\vec{r}_2| = -2a^2 \rightarrow |\vec{r}_2| = \frac{a^2}{|\vec{r}|} \rightarrow \vec{r}_2 = \frac{a^2}{|\vec{r}|^2}\vec{r}$$

$$q_2 = -\frac{a + |\vec{r}_2|}{|\vec{r}| + a} = -\frac{a + \frac{a^2}{|\vec{r}|}}{|\vec{r}| + a} = -\frac{a}{|\vec{r}|}\frac{|\vec{r}| + a}{|\vec{r}| + a} = \frac{-a}{|\vec{r}|}$$

Turns out all <u>other</u> points on sphere have $V = 0$ with this choice: $-\frac{a/|\vec{r}|}{4\pi|\vec{r}_2 - \vec{r}'|}$, $\vec{r}_2 = \frac{a^2}{|\vec{r}|^2}\vec{r}$

(Good vector review problem)

$\frac{-1}{4\pi|\vec{r}-\vec{r}'|}$ in spherical Laplace expansion

Simplification: Use angle between $\vec{r}, \vec{r}'$

($\alpha = \theta$ if $\vec{r}'$ along $+\hat{k}$)

Then, if using α, $m=0$ from azimuthal symmetry

$$\frac{-1}{4\pi|\vec{r}-\vec{r}'|} = \sum_\ell \left(C_\ell r^\ell + D_\ell{}^{-\ell-1}\right) P_\ell(\cos\alpha)$$

Let's pick expansion valid for away (large r):

$C_\ell = 0$

$$\frac{-1}{4\pi|\vec{r}-\vec{r}'|} = \sum_\ell D_\ell r^{-\ell-1} P_\ell(\cos\alpha)$$

- To match terms, expand $|\vec{r}-\vec{r}'|^{-1}$ as infinite series in r, θ

- Too hard! Equation holds for all α, so pick a simple one to fix D_ℓ; $\alpha = 0 \to \cos\alpha = 1$

$\alpha = 0$: $\vec{r}, \vec{r}'$ aligned, $|\vec{r}-\vec{r}'| = r - r'$ if $r > r'$

$$\frac{-1}{4\pi(r-r')} = \sum_\ell D_\ell r^{-\ell-1} \cancel{P_\ell(1)}$$ 1, by convention

$r > r'$: $\frac{1}{r-r'} = \frac{1}{r(1-\frac{r'}{r})} = \frac{1}{r}\sum_\ell \left(\frac{r'}{r}\right)^\ell$ converges

$$\frac{-1}{4\pi(r-r')} = \frac{-1}{4\pi r}\sum_\ell \left(\frac{r'}{r}\right)^\ell = \sum_\ell D_\ell r^{-\ell-1}$$

$$\frac{-1}{4\pi r}\sum_\ell \left(\frac{r'}{r}\right)^\ell = \sum_\ell D_\ell r^{-\ell-1}$$

$$\frac{-1}{4\pi}\sum_\ell \frac{r'^\ell}{r^{\ell+1}} = \sum_\ell D_\ell / r^{\ell+1}$$ $r^{\ell+1}$ terms match if

$$\frac{-1}{4\pi} r'^\ell = D_\ell$$

Full solution:

$$\frac{-1}{4\pi|\vec{r}-\vec{r}'|} = \sum_\ell D_\ell / r^{\ell+1} P_\ell(\cos\alpha)$$

$$= \sum_\ell \frac{-1}{4\pi} r'^\ell / r^{\ell+1} P_\ell(\cos\alpha)$$

$$= \frac{-1}{4\pi}\sum_\ell \frac{r'^\ell}{r^{\ell+1}} P_\ell(\cos\alpha) \qquad (r > r')$$

Integrating over this Green's form gives multipole expansion

$$V = \int \frac{-\rho(\vec{r}')}{\epsilon_0} G(\vec{r},\vec{r}')\, dV'$$

$$= \frac{1}{4\pi\epsilon_0}\int \rho(\vec{r}')\sum_\ell \frac{r'^\ell}{r^{\ell+1}} P_\ell(\cos\alpha)\, dV'$$

What if $r' > r$? Need alternate Green's expansion

$$G(\vec{r},\vec{r}') = \frac{-1}{4\pi|\vec{r}-\vec{r}'|} = \frac{-1}{4\pi}\sum_\ell \frac{r^\ell}{r'^{\ell+1}} P_\ell(\cos\alpha)$$

Similar to how we patched $t < t'$ and $t > t'$ in 1D (ODE) Green's functions