Last Time

- Minimization problems
- Vector functions

Today

- Differential operators
  - divergence, curl, etc.
- Curvilinear coordinates

Vector Functions of Many Variables

- Last time we looked at how vector functions behave under derivatives
- We also defined an operator, the gradient, which took scalar functions of many variables to a vector function (of those variables)
- There are more operators we would like to define on vector functions of many variables

Vector Functions of Many Variables

- Most of the time, we are interested in functions that map a space to itself
  - Linear example: square matrices
  - Gradient takes a scalar function of many variables to a vector function of many variables
  - Such maps called **vector fields**
  - We could also have extra variables in the domain

Divergence

- Divergence operator is a mode of differentiation that takes vector fields to scalar functions
  - Notation suggests we are taking “dot product” of gradient operator with vector field

- Interpretation of divergence: “spreading out” of field

Class Ex

- Find the divergence of $x^2 y^2 \hat{i} + y^2 z^2 \hat{j} + x^2 z^2 \hat{k}$

Laplacian

- Important combination operator: Laplacian
  - Gradient followed by divergence (takes scalar function back to scalar function)

$$\nabla^2 f = \nabla \cdot (\nabla f)$$

Curl

- Curl is a derivative operator that takes vector fields to vector fields
  - Like we are taking “cross product” of gradient operator with field
    - 3D only!

Class Ex

- Find curl of $x^2 y^2 z^2 \hat{i} + y^2 z^2 \hat{j} + x^2 z^2 \hat{k}$

Sum/Product Identities

- Gradient, divergence, curl distribute as you would think over sums
- Product rule works as you think as long as there is a scalar function
- Other identities more complicated

Combinations of Operators

- Already saw Laplacian
- Two other combinations are zero:

$$\nabla \times \nabla f = 0$$
$$\nabla \cdot (\nabla \times \vec{f}) = 0$$

- Rest more complicated

Curvilinear Coordinates

- Sometimes easier to analyze situations with symmetry with coordinates other than Cartesian (x,y,z)
  - We already saw this with some integrals
- If the transformation to new coordinates is non-linear, even basis vectors are functions
  - We usually make the coordinates independent, though, so they are orthogonal

Cylindrical Coordinates

- New coordinates are 2D polar and z axis

$$\rho = \sqrt{x^2 + y^2}$$
$$\phi = \tan^{-1}(y/x)$$
$$z = z$$

- Unit vectors:
  - Away from cylinder axis $\hat{e}_\rho = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$
  - Around cylinder $\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$
  - Along axis $\hat{e}_z = \hat{k}$
- Volume element: $dx\,dy\,dz = \rho\,d\rho\,d\phi\,dz$

Cylindrical Derivatives

- Because basis vectors not constant, derivative operators look different in cylindrical coordinates

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{e}_\phi + \frac{\partial f}{\partial z}\hat{e}_z$$

$$\nabla \cdot \vec{f} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho f_\rho) + \frac{1}{\rho}\frac{\partial f_\phi}{\partial \phi} + \frac{\partial f}{\partial z}$$

$$\nabla \times \vec{f} = \frac{1}{\rho}\begin{vmatrix} \hat{e}_\rho & \rho\hat{e}_\phi & \hat{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ f_\rho & \rho f_\phi & f_z \end{vmatrix}$$

Class Ex

- Find in cartesian and cylindrical coordinates, and compare, $\nabla^2 \rho$

Spherical Coordinates

- New coordinates:

$$r=\sqrt{x^2+y^2+z^2}$$
$$\theta=\cos^{-1}(z/r)$$
$$\phi=\tan^{-1}(y/x)$$

- Unit vectors:
  - Away from origin $\hat{e}_r=\sin\theta\cos\phi\,\hat{i}+\sin\theta\sin\phi\,\hat{j}+\cos\theta\,\hat{k}$
  - N/S on globe surface $\hat{e}_\theta=\cos\theta\cos\phi\,\hat{i}+\cos\theta\sin\phi\,\hat{j}-\sin\theta\,\hat{k}$
  - E/W on globe surface $\hat{e}_\phi=-\sin\phi\,\hat{i}+\cos\phi\,\hat{j}$
- Volume element: $dx\,dy\,dz=r^2\sin\theta\,dr\,d\theta\,d\phi$

Spherical Derivatives

$$\nabla f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{e}_\phi$$

$$\nabla \cdot \vec{f} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta f_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial f_\phi}{\partial \phi}$$

$$\nabla \times \vec{f} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ f_r & r f_\theta & r\sin\theta f_\phi \end{vmatrix}$$

Class Ex

- Find $\nabla^2 r$ in cartesian and spherical coordinates.

Next Time

- Vector integrals and integral theorems