Announcement

- Midterm 10/16 (in class, possible take-home portion)
  - Includes
    - integration tricks
    - Taylor series
    - linear algebra
      - including abstract stuff starting next week
    - vector calculus

Last Time

- Differential Operators
- Curvilinear coordinates

Today

- Potentials
- Integration with vectors
  - Line integrals
  - Surface integrals

Paths

- A 1D path in space can be represented by a position vector that depends on a parameter

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

- Often we want to know the contribution of a vector along a direction, e.g. $\vec{F} \cdot \hat{n}$
  - For a contribution along a path, this changes: only makes sense to do it for small increments

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

    - This is parallel to path at a point

Path Length

- We can find the size of this part of the path, integrate to get path length

Line Integrals

- We can integrate vector fields along paths using dot product $\int_C \vec{f} \cdot d\vec{r}$
  - This picks up part of vector along direction of path
    - ex: work $W = \int_C \vec{F} \cdot d\vec{r}$
  - For straight segments along a coordinate, this reduces to a simple coordinate integral

  - Otherwise, need to reduce to integral over one variable

Class Ex

- There is a force that is position-dependent, given by $\vec{F}=(x+y)\hat{i}+(y-x)\hat{j}$
  - Find the work done along the path (1,1) to (4,1), followed by (4,1) to (4,2), where each segment is straight
  - Find the work done along the curve $y^2=x$ from (1,1) to (4,2)

Line Integrals

- Problem: can’t simply integrate along each coordinate as if they’re independent unless only one coordinate changes along path
- General strategy: substitute coordinate variables with parameter, change variables, one integral left along parameter

Conservative Vector Fields

- If a vector field is a gradient of a function, then line integral is path-independent (depends only on endpoints)

- Check: if curl is zero, it can be written as gradient
- To find function (up to constant): compute line integral

Vector Potentials

- Sometimes vector field can be written as curl of another
  - Requirement: divergence zero

- Result is unique up to gradient of scalar function
  - To find: solve directly, but easiest to set one component zero

Example

- For $\vec{B} = (x^2 - yz)\hat{i} - 2yz\,\hat{j} + (z^2 - 2xz)\hat{k}$
  find **A** so that $\vec{B} = \nabla \times \vec{A}$

Surface Integrals

- You might also want how much of a vector field goes “through” a surface (component along perpendicular) $\int_S \vec{f}\cdot d\vec{A},\ d\vec{A}=\hat{n}\,dA$
  - Ex: Electric flux $\Phi_E=\int \vec{E}\cdot d\vec{A}$
  - Note: sometimes surface integrals written with double integral sign, even with dA

convention: outward

Defining Surfaces

- In terms of position directly:

$$\vec{r}(t,u)=x(t,u)\hat{i}+y(t,u)\hat{j}+z(t,u)\hat{k}$$

  - Normal is perpendicular to both surface directions

$$\vec{n}=\frac{\partial\vec{r}}{\partial t}\times\frac{\partial\vec{r}}{\partial u}$$

- Alternative, implicit:

$$f(x,y,z)=0$$

  - Normal: ∇f

Class Ex

- Find the flux of an electric field $\vec{E}=x\hat{i}$ through the hemisphere $x^2+y^2+z^2=a^2, z>0$

Unknown Area Elements

- If dA is unclear, project to xy plane, integrate over it

$$dA = \frac{dx\,dy}{\hat{n} \cdot \hat{k}}$$

Next Time

- Integral theorems
- Complex basics