Last Time

- Adjoint
- Diagonalization of operators
- Fourier Transform

Today

- First-order differential equations

Differential Equations

- General problem: we want to find a function given some relation among it and its derivatives, dependent variables

$$f(x, x', x'', \ldots, t) = 0$$
$$x(t) = ?$$

For now: *ordinary* differential equations, one independent variable

- Examples:
  - Force problems
  - Growth/evolution
  - Waves

First-Order Differential Equations

- Where to start: restrict problem to only first derivatives

$$f(x, x', t) = 0$$
$$x(t) = ?$$

- Maximum derivative called **order**

Main Strategy: Integrate

- Rearrange into an equation where you can antidifferentiate both sides
  - Ex: given $x'(t) = f(t)$
  - Ex: $x'(t) = a\, x(t)$

Class Ex

- An object, mass m, is falling through a fluid with resistive force kv. Find its velocity as a function of time, given initial velocity v_0.

General First-Order Solution

- Can broaden to general first-order equation with constant coefficients

$$\frac{dx}{dt} = a\,x + b$$

$$x = C\,e^{at} - \frac{b}{a}$$

Linear Equations

- Linear differential equations are ones where the operator equation L[x] = 0 is a linear operator
  - (repects linear rules for combining functions)
- Linear differential equations always have a solution
  - n independent solutions for order n, general solution a linear combination of them

General Linear First-Order Solution

- Use method of integrating factor on general equation

$$\frac{dx}{dt} = p(t)x + q(t)$$

  - multiply both sides by helper function μ(t) so a change of function will produce constant coeff result

More General First-Order Equations

- Non-linear equations not guaranteed unique solutions
- Sometimes you can get lucky and solve by separating dependent and independent variables

Class Ex

- Find an implicit relation between x and t for

$$\frac{dx}{dt} = \frac{t^2}{1 - x^2}$$

Homogeneous* Equations

- *Homogeneous in the algebraic sense—ratio of polynomials of two variables with all same order

$$\frac{dx}{dt}=\frac{f(x,t)}{g(x,t)}=h(x/t)$$

  - Homogeneous differential equations mean something else entirely later
  - To solve: divide both polynomials by t^n, now have differential equation in new variable, x/t

Class Ex

- Solve $dx/dt = (3x^2-xt)/(x^2+t^2)$
  (you may stop at the integral)

Bernoulli Equation

- There’s a trick to some simple non-linear equations of the form

$$\frac{dx}{dt} = p(t)\,x + q(t)\,x^n$$

- Make the substitution $u = x^{1-n}$
  - Now of the form

$$\frac{du}{dt} = (1-n)\,p(t)\,u + (1-n)\,q(t)$$

Next Time

- Some more non-linear techniques
- Linear second-order equations