# Today

- Methods useful for inhomogeneous equations
  - Laplace transform
  - Variation of parameters
  - Green's functions
  - Variation of parameters/Green's functions requires knowing homogeneous solutions, Laplace does not

# Laplace Transform

- Idea: convert ODE from original variable to another space where ODE looks simpler, solve there algebraically
- **Laplace transform**: “non-imaginary Fourier transform” $\bar{x}(s)=\int_0^\infty dt\, e^{-st}\, x(t)$

# Class Ex

- Find the Laplace transform of f’(t) in terms of the transform of f(t).

# Laplace Transform of Derivatives

- Generally, transform of nth derivative:

$$s^n \bar{x}(s) - s^{n-1} x(0) - s^{n-2} x'(0) - \ldots$$

- Transform very useful for initial value problems where given conditions are x(0), x'(0), etc.
- Most useful for constant-coefficient ODEs where transforms of derivatives are simple

# Completing the Solution

- The Laplace transform of a constant coefficient ODE will become an algebraic equation in s
- We want the solution in the original space—need inverse transform
- Computing inverse more difficult, unlike Fourier—requires math we haven't reviewed
- For now: create dictionary of common transforms, work backwards

# Class Ex

- Find the Laplace transform of $e^{kt}$.

# Common Transforms

- Transforms are usually only defined in part of s space (s > 0 unless stated)

| | | |
|---|---|---|
| $c$ | ➔ | $c/s$ |
| $ct^n$ | ➔ | $cn!/s^{n+1}$ |
| $\sin kt$ | ➔ | $b/\left(s^2+k^2\right)$ |
| $\cos kt$ | ➔ | $s/\left(s^2+k^2\right)$ |
| $e^{kt}$ | ➔ | $1/\left(s-k\right), s>k$ |
| $t^n e^{kt}$ | ➔ | $n!/\left(s-k\right)^{n+1}, s>k$ |
| $e^{kt}\sin at$ | ➔ | $a/\left(\left(s-k\right)^2+a^2\right)$ |
| $e^{kt}\cos at$ | ➔ | $\left(s-k\right)/\left(\left(s-k\right)^2+a^2\right)$ |
| $\delta\left(t-t'\right)$ | ➔ | $e^{-st'}$ |

# Example

- Solve the ODE $x''-3x'+2x=e^{-t}$
  if x(0) = 0, x'(0) = 2 using the Laplace transform.

# Variation of Parameters

- There is a more general method of getting particular solutions than guessing the form
- Assume form of particular solution is a combination of homogeneous solutions
  - 2nd order: $a(t)x''+\ldots=f(t), x_p=k_1(t)x_1(t)+k_2(t)x_2(t)$
  - Get k to produce inhomogeneous piece
    - Choose $k_1'x_1+k_2'x_2=0, k_1'x_1'+k_2'x_2'=f(t)/a(t)$
  - System of equations for k', integrate to get k and particular solution

# Class Ex

- Solve $x'' + x = 1/\sin t$ for the boundary conditions $x(0) = x(\pi/2) = 0$.

# Green’s Functions

- Goal: to solve all inhomogeneous for given differential equation, by finding “root” solution
- To solve $L[x] = f(t)$, solve $L[G(t,t')] = \delta(t-t')$
  - Get x for $f(t)$ by integrating $G(t,t')f(t')$

# Properties of G

- Highest derivative of G ($2^{nd}$ for $2^{nd}$ order) must produce delta function spike at t=t'
- So next-highest ($1^{st}$ for $2^{nd}$ order) must have a discontinuity at t=t' (produces step function)
  - Write G in two pieces: t' < t, t' > t
  - $2^{nd}$ order: enforce $dG/dt(t,t+\varepsilon) - dG/dt(t,t-\varepsilon) = 1$
- Want lower derivatives to be continuous
  - Enforce $G(t,t+\varepsilon)=G(t,t-\varepsilon)$

# Example

- Solve $x'' + x = \delta(t-t')$ for the boundary conditions $x(0) = x(\pi/2) = 0$.

# Example

- Re-solve $x'' + x = 1/\sin t$ using the Green's function found earlier.