Last Time

- Series methods

Today

- Sturm-Liouville theory

Hermitian Operators

- We know if an operator is hermitian for a given inner product, we can diagonalize it with a basis change
- So, given a hermitian linear differential operator $L[x] = \lambda x$, if we find the eigenfunctions x that diagonalize the operator:
  - we solve the differential equation for all λ
  - the eigenfunctions are a complete set of orthogonal functions for the corresponding inner product

Sturm-Liouville Equations

- Some second-order equations are hermitian up to boundary terms
  - * switching to x as dependent variable, y as independent for this subject, so we can take inner products to be x integrals

$$[\frac{d}{dx}(p(x)\frac{d}{dx})+q(x)]\,y(x)=\lambda\,y(x)$$

$$L[y]=\lambda\,y$$

Making it Hermitian

- Two ways to get a hermitian result:
  - Boundary terms accidentally satisfied by $p(a)=p(b)=0$
  - Boundary conditions
    - $y(a)=y(b)=0$, define inner product over interval (a,b)
      - Called **Dirichlet** boundary condtions, can sometimes make a change of variable to achieve this
    - $y'(a)=y'(b)=0$: define inner product over interval (a,b)
      - Called **Neumann** boundary condtitions

Class Ex

- The operator $\frac{d}{dx}\left(\left(1-x^2\right)\frac{d}{dx}\right)$ is hermitian for what bounds on the inner product?

Examples of Hermitian Equations

- Harmonic $[\frac{d}{dx}(\frac{d}{dx})+0]\,y(x)=\omega^2\,y(x)$
- Legendre $[\frac{d}{dx}((1-x^2)\frac{d}{dx})+0]\,y(x)=l(l+1)\,y(x)$
- Associated Legendre

$$[\frac{d}{dx}((1-x^2)\frac{d}{dx})-m^2/(1-x^2)]\,y(x)=l(l+1)\,y(x)$$

Putting Equations into S-L Form

- Sometimes we can “fix” the ODE by multiplying by a factor, making it hermitian
- Changes necessary inner product by requiring a weighted integral

Examples of Weighted Hermitian

- Hermite $[\frac{d}{dx}(e^{-x^2}\frac{d}{dx})+0]\,y(x)=2\,\nu e^{-x^2}\,y(x)$
- Laguerre $[\frac{d}{dx}(xe^{-x}\frac{d}{dx})+0]\,y(x)=\nu e^{-x}\,y(x)$
- Bessel $[\frac{d}{dx}(x\frac{d}{dx})-\nu^2/x]\,y(x)=\alpha^2 x\,y(x)$

So What?

- Hermitian operators appear frequently in physics, but...
- We still need to solve the hermitian (homogeneous) ODEs using series methods, etc.
- But, we can solve inhomogeneous by transforming to the diagonal space in the eigenfunctions

Green’s Functions for S-L

- There is a direct method to get Green’s functions from eigenfunctions

Class Ex

- Find a Green's function for $L[y] = \frac{d^2 y}{dx^2} + \frac{1}{4} y$

  with y(0) = y(π) = 0 by verifying and using that it is a hermitian operator.

Class Ex

- Solve $\frac{d^2 y}{dx^2} + \frac{1}{4} y = x$

  (same BC as before)

Next Time

- PDEs