Last Time

- 1^{st} Order PDEs
- Characteristics

Today

- Inhomogeneous PDEs
- Higher-order behavior
- Separation of Variables

Inhomogeneous Equations

- Like ODEs, solve homogeneous first, add particular solution that matches the RHS

Class Ex

- Find the general solution to $\partial_t u - c\,\partial_x u = x$

Higher-order

- The nature of reasonable boundary conditions depends crucially on the form of the PDE
- We'll take a look at some second-order and note allowable boundary conditions that allow unique solutions

Wave Equation

- Wave equation: $\partial_t^2 u - c^2 \nabla^2 u = 0$
- 1D version (in space, really 2D total): $\partial_t^2 u - c^2 \partial_x^2 u = 0$
- Solutions by factorization

- Two independent function solutions
  - 2nd order generally have two independent solutions, can build with linear combination
- Characteristics intersect—function can only be determined from initial conditions along a nice line

Laplace Equation

- Laplace Equation: $\nabla^2 u = 0$
- Similar solutions to wave equation, but “complex” factorization

- Solutions do not propagate along characteristics
- Appropriate boundary conditions more complicated

Diffusion/Heat Equation

- Missing 2^{nd} derivative in time $\partial_t u = \kappa \partial_x^2 u$
- No characteristics
- Generally solutions only stable under propagation in one direction (past/future, not both)

Types of Boundary Conditions

- Cauchy: function defined on subspace, with normal derivative (if second order)
  - analog to initial conditions of ODEs
- Dirichlet: function defined on boundary enclosing solution space
- Neumann: function's normal derivative defined on boundary enclosing solution space

Classes of 2^{nd} Order

- Nature of boundary conditions that produce good solutions dependent on how 2^{nd} order equation would be “factorized”

$$\left(A\,\partial_{tt}+B\,\partial_{tx}+C\,\partial_{xx}+\ldots\right)u=0$$

  - B^2-4AC > 0: real roots, **elliptic**
    - ex: Poisson
  - B^2-4AC = 0: repeated roots, **parabolic**
    - ex: heat
  - B^2-4AC < 0: complex roots, **hyperbolic**
    - ex: wave

Allowed 2^{nd} Order BC

Class	BC with unique solution
hyperbolic	Cauchy, open region only
parabolic	Dirichlet/Neumann, open region, solution converges only in one direction
elliptic	Dirichlet/Neumann, closed boundary of region

Separation of Variables

- Not always easy/possible to analyze characteristics
- New strategy: guess a solution of a form where variables can always be separated
  - By itself, this will not be the *general* solution
  - Doesn’t always work, but useful for linear PDEs

Separation Strategy

- Assume solution is a product of single-variable functions
- Insert into PDE
- Divide PDE by solution
- Result should be an equation with terms of only one variable
- Only possibility: every term is a constant
  - called **separation constant(s)**
    - Will need n-1 constants for n variables, will be relations between

Next Steps

- Every separated term is now an ODE containing the separation constant
- Solve using usual ODE methods
- Multiply solutions to get PDE solution

Example

- Find a separation solution to $\partial_t u = c\,\partial_x u$

Building Solutions

- We only got a single solution: one that matched the separation constant
- More general solutions can be built with a linear combination of these solutions for different separation constants
- The separation constants tie together the independent single-variable solutions
  - Play similar role to characteristic lines

Applying Boundary Conditions

- Separated solutions are ideal for boundary conditions that lie along variable axes
- In fact, if you successfully separated:
  - Remaining ODEs are eigenvalue equations
  - Part of boundary conditions will enforce specific (often integer) eigenvalues
  - If remaining ODEs are hermitian, solution space is complete, therefore PDE solution will exist in combination of separated solutions

Class Ex

- Solve the 3D quantum mechanical infinite square well, where the particle is confined to be in $0<x<a$, $0<y<b$, $0<z<c$.

$$\frac{-\hbar^2}{2m}\nabla^2\Psi=i\hbar\frac{\partial\Psi}{\partial t}$$

Next Time

- Separations in non-Cartesian coordinates