# Last Time

- Separation of Variables

# Today

- Separation examples in non-Cartesian coordinates

# Other Coordinates

- The separation strategy is ideal when the boundary conditions are “separated” along a subset of the coordinates
  - Guaranteed to get the solution in many situations due to completeness of ODE eigenvalue solutions
- Most important PDEs in physics involve the laplacian with possible time derivatives—nice 2nd order PDEs with constant coefficients
- However—boundary conditions and useful coordinate systems are often non-Cartesian

# 2D Laplacian

- 2D Laplacian: $\nabla^2 = \partial_x^2 + \partial_y^2$
- We could solve Laplace's equation $\nabla^2 u = 0$ easily by separation in Cartesian coordinates
- What if we want solution in polar? (Very needed if boundary is a circle!)
  - Laplacian in polar: $\nabla^2 = \frac{1}{\rho}\partial_\rho(\rho\partial_\rho) + \frac{1}{\rho^2}\partial_\phi^2$
  - Separation still possible in these coordinates with a little extra work

# 2D Laplace Polar ODE Solutions

- Angular solution still simple, but geometry provides implicit boundary condition: must be periodic in $\varphi$

- Radial ODE not obvious since non-constant coefficients, but we can try power series

# Class Ex (Review?)

- Find the recurrence relation to for the ODE

$$\rho^2 P'' + \rho P' - \lambda P = 0$$

# More on 2D Polar Laplace

- Radial ODE singular at zero, need to be careful near there
- Can eliminate half of radial solutions with “physical” boundary condition argument: solution should not blow up too badly at zero
- Different form of solution if $\lambda=0$

# Class Ex

- A drum of radius a has a fixed displacement along its rim perpendicular to the drum face that varies with angle: $u(a,\phi)=\epsilon\sin(\phi+2\sin 2\phi)$ Find the displacement for the entire drumskin surface.
  - Hint: the drumskin is elastic, and obeys the 2D wave equation, or when fixed, the 2D Laplace equation

# Applying Boundary Conditions

- Observation: boundary/initial conditions will often “pick out” the allowed eigenvalues when doing separation of variables/eigenfunction expansion, determine rest of solution

# 3D Laplace, Spherical Coordinates

- Laplace's equation, spherical polar:

$$[\frac{1}{r^2}\partial_r\left(r^2\partial_r\right)+\frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\right)+\frac{1}{r^2\sin^2\theta}\partial_\phi^2]u=0$$

# Class Ex

- Separate the 3D spherical polar Laplace equation, solving the φ equation and setting up the ODE in θ.

# Associated Legendre Equation

- Associated Legendre Equation:

$$\left[\frac{d}{d\cos\theta}\left(1-\cos^2\theta\right)\frac{d}{d\cos\theta}+\lambda-\frac{m^2}{1-\cos^2\theta}\right]\Theta=0$$

- Same as Legendre equation with m=0 (x=cosθ)
- Solutions: built from Legendre solutions

$$P_l^m(x)=\left(1-x^2\right)^{m/2}\frac{d^m P_l}{dx^m},\lambda=l\left(l+1\right)$$

  - (Tables of these available many places)
  - Like Legendre polynomials, mutually orthogonal
  - Solutions good at poles require |m| ≤ l, l integer

# Spherical Harmonics

- Combination of φ/θ solutions, normalized, called **spherical harmonics**

$$Y_l^m(\theta,\phi)=(-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,e^{im\phi}P_l^m(\cos\theta)$$

- Orthonormal, complete set of functions on angular space in spherical coordinates
- Any PDE with spherical symmetry separates this way to produce spherical harmonics
- Also easy to look up particular $Y_l^m$

# The Radial Part

- Radial equation: $r^2 R'' + 2r R' - \lambda R = 0, \lambda = l(l+1)$

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- Radial equation: $r^2 R'' + 2r R' - \lambda R = 0, \lambda = l(l+1)$
- Solution: $R_l(r) = C_l r^l + D_l r^{-l-1}$

# Class Ex

- An uncharged conducting sphere is placed in an initially uniform external electric field of magnitude E. Find the potential everywhere outside the sphere.
  – Hints:
    - Laplace's equation holds *outside* the sphere where there is no charge.
    - Perfect conductors are equipotentials—can use as a boundary condition
    - Behavior of field at infinity is a boundary condition

# Next Time

- Other PDE examples?
- Green’s functions?