Last Time

- Non-cartesian Laplace separations

Today

- Green's functions for PDEs (Laplace/Poisson example)
  - Method of images
- Multipole expansions

Inhomogeneous PDEs

- Much like ODEs, can have PDEs with extra function on RHS
- Classic example: Laplace -> Poisson

$$\nabla^2 u = 0 \rightarrow \nabla^2 u = \rho(\vec{r})$$

- Occasionally can solve directly if RHS is function of one variable, use usual particular solution fix with that variable’s ODE
- More general method: solve for delta RHS, get other solutions by integration (Green’s function solution)

$$\nabla^2 G = \delta(\vec{r} - \vec{r}\,')$$

ODE vs PDE (Laplace) Green’s Functions

- Integral of highest derivative gives delta
  - ->discontinuity of 1 in next highest derivative around t=t’
- G continuous at t=t’
- G depends on BC, apply to fix form

- Volume integral of laplacian gives delta
  - ->surface integral of normal derivative just outside r=r’ is 1
- ???
- Hard to find G for given BC
  - Find for easiest (zero) BC, repair later for specific BC

Example

- Solve the Poisson equation $\nabla^2 u = \delta^3(\vec{r} - \vec{r}\,')$ subject to the boundary condition of vanishing at infinity. (Find the Green's function for the laplace operator under that BC.)

Green's Functions for New BC

- If you have the "fundamental" Green's function (vanishes at infinity) one can get other zero BC by adding a laplace solution so that total obeys BC
- Simplest way is often adding Green's functions at other spots!
  - Original Green's has r inside solution volume
  - New ones have r outside—since r is never r', solves laplace, still good Green's function for original volume
  - Adjust position and size of copies to get G = 0 on boundary
  - Called **method of images**

Class Ex

- A grounded infinite plane ($z=0$) has a line of charge, density λ, placed above it at $z=a$ along the x-axis, from $-L$ to L. Find the potential for $z > 0$.

Spherical Images

- If spherical surface is zero boundary, can still use method of images with a single “charge”
- Class Ex: For a charge outside the sphere (radius a), find location and size of charge inside that cancels the potential at a point in between and a point opposite
  - Can prove this sets potential to zero everywhere on sphere

Nonzero Boundary Conditions

- If the desired BC are nonzero, but one has the zero Green's function, can still get solution:

$$u(\vec{r}) = \int_V dV' G(\vec{r}, \vec{r}\,')\rho(\vec{r}\,') + \int_S f(\vec{r}) \frac{\partial G}{\partial n} dS'$$

BC: u(r) on S

Green’s Function Expansions

- Separated solutions end up as expansions in products of eigenfunctions
- Green’s functions solve homogeneous PDE if r is outside solution volume
  - They have an expansion in terms of homogeneous solution

Example

- Expand the fundamental laplace Green’s function in spherical coordinates.

Multipole Expansion

- Poisson solution built using expanded Green’s function called multipole expansion

- Interpretation: first term is potential of point charge, second is potential of dipole, third quadrupole...
  - First nonzero term is good approximation of potential

Further Applications of Green's

- Wave equation $\partial_t^2 u = c^2 \nabla^2 u$
  - EM waves with sources
- Helmholtz equation $\nabla^2 u = k^2 u$
  - Scattering of time-independent wavefunctions
    - Approximate potential part using asymptotic "incoming" wavefunction (Born approximation)

Reminders

- Final exam Tuesday 10am
  - Emphasis on material after midterm
    - ODEs
    - PDEs
- HW extended to tomorrow (turn in Keen 513)