Last Time

- Gauss-Jordan inversion
- Eigenvectors & eigenvalues
- Change of basis
  - Similarity transformations

This Time

- More on similarity
- Diagonalizing matrices
- Meaning of the determinant
- Special orthogonal matrices

Changing Basis

- Matrices can be used to transform the coordinates of a vector from one basis to another

  - **Similarity transformation**: $A' = S^{-1} A S$
    - New basis vectors are columns of S

Class Ex

- Find the form of the matrix $\begin{pmatrix} 2 & -1 \\ 2 & 5 \end{pmatrix}$

  in the new basis $\vec{e}_1{}' = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{e}_2{}' = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Similarity Properties

- Similarity transformations:
  - Keep eigenvalues same
  - Keep determinant the same

Class Ex

- Show that the similarity transformation of A^n is the similarity transformation of A, to the power n.

Diagonalization

- Special basis: use eigenvectors of matrix
  - In new basis, this makes matrix diagonal
    - Eigenvalues are diagonal entries

- Similarity transformation: change basis to one where matrix is diagonal, change back

Uses of Diagonalization

- Change of basis allows linear operator that mixes up coordinates to separate them out again in new basis
  - Solve each coordinate “separately”
- Diagonal matrices have extremely simple properties
  - Powers easily

Ex (Coupled Oscillators)

- Find the **characteristic frequencies**, the frequencies of oscillation allowed by the coupled system, for two pendula of mass m, length L, connected by a spring of constant k, in the small angle approximation. The spring is unstretched when the masses hang vertically.

Matrix Miscellaneous

- Array notation versus matrix components
  - Matrix in symbol form is not commutative, but elements in equations are if careful with indices

- More matrix formulas

$$(A\,B)^T = B^T A^T$$
$$(A\,B)^{-1} = B^{-1} A^{-1}$$

Row Vectors, Etc.

- We write vectors as n x 1 matrix (column vector)
- Transpose of this is row vector, 1 x n matrix

$$(\vec{u})^T \vec{v} = \vec{u} \cdot \vec{v}$$ (usual orthonormal basis)

Meaning of Determinant

- 3D formula:

$$\vec{a}\cdot(\vec{b}\times\vec{c})=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

  - Volume of parallelipiped spanned by three vectors
- *Any* determinant of n x n is “volume” spanned by n vectors
- In a similarity transformation S, volume is scaled by determinant of S
- Vanishing determinant means at least some direction maps to zero
  - Bonus terminology: **rank** of matrix is number of independent directions *not* mapped to zero

(Special) Orthogonal Matrices

- **Orthogonal** matrix: made up of orthogonal (and normalized) vectors
  - Basis change with determinant +/-1
  - $A^{-1} = A^T$
  - If determinant 1, **special** orthogonal matrix, aka rotation

Class ex

- Given $\theta = \cos^{-1}\left(\hat{e}_1 \cdot \hat{e}_1{}'\right)$
  find the 2X2 special orthogonal matrix describing the transformation to the new coordinate system

Class Ex

- If A is symmetric, show $R^{-1}AR$ is symmetric for R orthogonal.

The Trace

- The **trace** is the sum over diagonal components $Tr(A)=\sum_i A_{ii}$

- Often comes up when summing over things of the form $\sum_i \vec{u}_i \cdot (A\vec{u}_i)$

ex: orthonormal set

Trace Properties

- Trace of multiple matrices is cyclic
- $Tr(A^T) = Tr(A)$
- Is sum of eigenvalues $Tr(A) = \sum_i \lambda_i$
  - Is invariant under similarity transformation

Next Time

- Vector calculus basics