Today

- New major topic: multivariate calculus
- Today: functions of more than one variable
  - Partial derivatives
  - Multivariate chain rule
  - Multivariate Taylor series
  - Multidimensional integrals

Multivariable Functions

- First, we’ll look at functions from multiple variables to a single value
  - Think of these as functions from a vector to a scalar

Differentiating

- 1D calculus: derivative is rate of change
- 2D, 3D, etc.: More than one variable to change!
  - Change one at a time: partial derivative (hold others constant)

$$\frac{\partial f}{\partial x}$$

    - Partial derivatives commute (if they exist)

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\frac{\partial f}{\partial y}=\frac{\partial^2 f}{\partial x \partial y}$$

Total Derivative

- In principle, can change any variable
- Total derivative: change in f for any given small change in other variables

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \ldots$$

Chain Rule

- What if multiple variables are functions of another? $f\left(x(t), y(t), \ldots\right)$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \ldots$$

Multivariable Taylor Series

- Taylor series is a mixture of expansions of each variable

Class Ex

- Find the Taylor expansion of exp(xy) about x=y=0, up to second derivative terms.

Integrating Multiple Variables

- Can integrate multiple variables, one after the other
  - Order doesn’t matter, like differentiation, unless limits depend on other integration variables
  - If integration variables are coordinates, interpretation is integrating over area or volume
    - Usual orthogonal coordinate system: $dA = dxdy$, $dV = dxdydz$

Class ex

- Find the moment of inertia of a uniform 45/45/90 degree triangle with short side length L, about an axis perpendicular to it through one of the 45 degree corners.

$$I = \int r^2 \, dm$$

$$dm = \lambda \, dl \, , dm = \sigma \, dA \, , dm = \rho \, dV$$

The Jacobian

- Can do change of variables like 1D integrals, but need a nontrivial factor, called **Jacobian**

$$dx\,dy\ldots = |J|\,dx'\,dy'\ldots$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} & \cdots \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix}$$

Class Ex

- By changing coordinates x = r cosθ, y = r sinθ, integrate

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{-x^2-y^2}$$