Last Time

- Functions of more than one variable
  - Partial derivatives
  - Taylor series
  - Integrals

Today

- Minimization problems
  - Lagrange Multipliers
- Vector functions

Minimization (or Maximization)

- 1D calculus: derivative must be zero at local extrema
- Multidimensional calculus: *all* derivatives must be zero

  - Minimum, maximum, or mixed (**saddle point**) depends on second derivatives

Gradient

- Set of all derivatives, assembled as vector: **gradient**
- Gradient takes scalar multivariable functions to vector-valued multivariable functions
- Magnitude of gradient is maximum rate of change of function
- Direction of gradient is direction of fastest increase
  - opposite fastest decrease

Class Ex

- Find the minimum, maximum, and saddle points of

$$f(x, y) = x^3 \exp(-x^2 - y^2)$$

Nature of Extrema

- To be a true minimum or maximum, second derivatives in *every* direction (not just each coordinate) should be same sign

- If function had no mixed partial derivatives, would simplify to checking each coordinate
  - Find better coordinates by diagonalizing matrix of second derivatives
    - Extremum if all eigenvalues are same sign

Class Ex

- Check the nature of the extrema of the previous example.

Constraints

- Sometimes you want to find minima (or maxima) of a function, but only for a limited domain
- If this domain is a subset of lower dimension (ex: intersection with another surface), eliminate a coordinate and proceed

Class Ex

- Prove the engineer’s result from the joke on the first day of class, that for a given amount of fence L, the area enclosed by a rectangle is maximized if the sides are equal length.

Lagrange Multipliers

- Eliminating variables in constraints can produce very ugly functions
- Lagrange multiplier: add an extra variable and part of the function that enforces the constraint
  - Instead of minimizing f after eliminating with constraint g, minimize $f+\lambda g$

Class Ex

- Do the rectangle exercise again with a Lagrange multiplier

Vector-valued Functions

- As we saw in linear algebra, we can have functions that produce multiple output (vectors)
- For now: vector functions of one variable
  - Ex: curves in space, time-dependent vector quantities

Basic Properties of Vector Functions

- Addition/subtraction/differentiation/integration are component-by-component

- Differentiation of products works as you would expect

Class ex

- Prove the centripetal acceleration rule by differentiating r^2 = const for a circle.

Directional Derivative

- Given a direction (unit vector) can project gradient on that direction to get **directional derivative**

$$\hat{n} \cdot \vec{\nabla} f$$

Next Time

- Multivariable vector functions
- More vector operators (divergence, curl, etc.)
- Curvilinear coordinates?