**Constant Acceleration in Special Relativity**

David Wright

Tallahassee, Florida

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The following is a derivation of constant acceleration time transformation equations starting with the Lorentz Transformations. These equations can be found in several sources on Special and General Relativity (Misner, Thorne, and Wheeler’s Gravitation for example, p167).[[1]](#footnote-1)

Consider the Lorentz Transformations[[2]](#footnote-2) for a point  in rest frame and moving frame, where  moves with velocity  in the positive  direction in. The Lorentz Transformations from rest frame to moving frame is stated first, followed by the Inverse Lorentz Transformations (). The differentials of these transformations are found and rearranged to find the time transformation equations between frames where the point accelerates from rest at a constant rate.

The **Lorentz Transformations** for point  where  moves with speed  in the positive  direction,  is the rest frame time,  is the speed of light, and,

|  |  |
| --- | --- |
|  | (1a)  (1b)  (1c)  (1d) |

The **Inverse Lorentz Transformations** for point,

|  |  |
| --- | --- |
|  | (2a)  (2b)  (2c)  (2d) |

Taking the differentials of (2a) and (2d),

|  |  |  |
| --- | --- | --- |
|  |  | (3a) |
|  |  | (3b) |

where  is the speed of  in .

Dividing (3a) by (3b) to find the speed  of the point in,

,

|  |  |  |
| --- | --- | --- |
|  |  | (4) |

Differentiating  to find the acceleration  of the point in,

,

Substituting  in the above,

,

|  |  |  |
| --- | --- | --- |
|  |  | (5a) |

The acceleration of point  in ,  is held constant; to emphasize this and to simply the writing, the acceleration of the point in the moving frame will be represented with “*a*.” Attach an ideal clock to the point  accelerating in, while  and  momentarily have the same *instantaneous* speed, *v*. Summarizing[[3]](#footnote-3),

, , and , from [(4)](#Equation4).

Therefore (5a) can be written,

|  |  |  |
| --- | --- | --- |
|  | , | (5b) |

Integrating (5b),

|  |  |  |
| --- | --- | --- |
|  | , | (5c) |

Rearranging (5c),

,

|  |  |  |
| --- | --- | --- |
|  | , | (5d)[[4]](#footnote-4) |

Rearranging again to find another useful form of ([5c](#Equation5c)),

,

|  |  |  |
| --- | --- | --- |
|  |  | (5e) |

Integrating ([5e](#Equation5e)) to find the time transform equations with constant acceleration,

,

|  |  |  |
| --- | --- | --- |
|  | **Time Transform with Constant Acceleration** | () |

Finding the inverse of ([](#Equation1alpha)),

,

|  |  |  |
| --- | --- | --- |
|  | **Inverse Time Transform with Constant Acceleration** | () |

where  is the time within  (the rest frame),  is the acceleration of in the moving frame,  is the speed of light, and  is time within the moving frame .

REFERENCES

Acosta, V., Cowan, C. L., & Graham, B. J. (1973) *Essentials of Modern Physics*. Harper & Row: New York.

Arzeleiès, Henri (1972) *Relativistic Point Dynmaics* . trans. by P. W. Hawkes, Pergamon P: Oxford.

DeWitt, Bryce (2011) *Bryce DeWitt’s Lectures on Gravitation, Edited by Steven M. Christensen, Lecture Notes in Physics, 826.* Springer-Verlag: Berlin-Heidelberg.

Einstein, Albert (1952) *The Principle of Relativity (On the Electrodynamics of Moving Bodies, trans. by W. Perrett and G. B. Jefferey from “Zur Elecktrodynamik bewegter Körper,” Annalen der Physik, 17, 1905)*, Dover: New York.

(Einstein, Albert, 1918, *Dialogue about Objections to the Theory of Relativity/Dialog über Einwände gegen die Relativitätstheorie* . Naturwissenschaften, 6, 697–702.)

French, A. P. (1968, 1966) *Special Relativity, The M. I. T. Introductory Series.* W. W. Norton & Company Inc.: New York.

Marder, L. (1971) *Time and the Space-Traveller* . U of Pennsylvania P: Philadelphia.

Møller, C. (1962) *The Theory of Relativity* . Oxford UP: Amen House, London.

Schlegel, Richard (1968) *Time and the Physical World*. Dover Publications, Inc. (1961, Michigan State University Press): New York.

LINKS OF INTEREST

[Susskind Lectures on Relativistic Kinematics](http://www.lecture-notes.co.uk/susskind/special-relativity/lecture-6/relativistic-kinematics)

[George Smoot on Relativity: Physics 139 Relativity, UC Berkeley](http://aether.lbl.gov/www/classes/p139/homework/homework.html)

[The Original Usenet Physics FAQ: The Relativistic Rocket](http://math.ucr.edu/home/baez/physics/Relativity/SR/rocket.html)

[scienceworld.wolfram: "Proper Time"](http://scienceworld.wolfram.com/physics/ProperTime.html)

[scienceworld.wolfram: "Velocity Four-Vector"](http://scienceworld.wolfram.com/physics/VelocityFour-Vector.html)

[Leonard Susskind on Special Relativity (YouTube)](https://www.youtube.com/watch?v=BAurgxtOdxY)

[Restarting the LHC: Why 13 Tev?](http://home.web.cern.ch/about/engineering/restarting-lhc-why-13-tev)

NASA – Kepler-452b-Earth-Like-Planet

Summary of equations used in <http://www.zitterbug.net/future/future815.html>

|  |  |  |
| --- | --- | --- |
| **STARSHIP TIME TO EARTH TIME:** |  | [(8a)](#Equation8a) |
| **EARTH TIME TO STARSHIP TIME:** |  | [(8b)](#Equation8b) |
| **EARTH TIME MINUS STARSHIP TIME:** | The Newton-Raphson method is used to solve for *T’* given *T* or *T* given *T’.* |  |
| **MAXIMUM SPEED:** |  | [(5d)](#Equation5d) |
| **FARTHEST DISTANCE:** |  |  |
| **EARTH TIME TO DESTINATION:** |  | [(7b)](#Equation7b) |
| **SHIP TIME TO DESTINATION:** |  | [(6b)](#Equation6b) |

Summary of useful forms of ([5c](#Equation5c))

|  |  |  |
| --- | --- | --- |
|  |  | (5c)  (5d)  (5e) |

1. Misner, Charles W.; Thorne, Kip S.; Wheeler, John Archibald (1973) *Gravitation*. W. H. Freeman & Company: New York, P163-167; Marder, L., (1971) *Time and the Space-Traveller*. University of Pennsylvania Press: Philadelphia, P183. [↑](#footnote-ref-1)
2. There are several very good introductions and derivations of the Lorentz Transformations (see [References](#References)). [↑](#footnote-ref-2)
3. It might help to recall that a point moving in simple harmonic motion can have a non-zero instantaneous acceleration and instantaneous speed of zero. [↑](#footnote-ref-3)
4. Note that in terms of the ship’s time (a.k.a. *proper time,*), equation [(5d)](#Equation5d) becomes. [↑](#footnote-ref-4)