4

Energy, Momentum and Mass-Shell

4.1 Introduction

The aim of this Chapter is to explain the mechanical properties of elementary particles that will form the basis of much that we shall be discussing. In particular, it is necessary to have a good understanding of momentum and energy, and, for a single particle, the relation between the two, called the mass-shell relation. Energy and momentum are important concepts because of two facts: first they are, in the context of quantum mechanics, enough to describe completely the state of a single free particle (disregarding internal properties such as spin and charge), and second, they are conserved. For energy this is well known: for any observable process the initial energy equals the final energy. It may be distributed differently, or have a different form, but no energy disappears. If we burn wood in a stove the chemical energy locked in the wood changes into heat that warms the space where the stove is burning; eventually this heat dissipates to the outside, but does not disappear. This is the law of conservation of energy. Similarly there is a law for conservation of momentum and we shall try to explain that in this section for simple collision processes.

The fact that a description of the state of a particle in terms of its energy and momentum is a complete description is very much at the heart of quantum mechanics. Normally we specify the state of a particle by its position and its momentum at a given time:

where, when and how does it move. Momentum[a] is a vector, meaning that it has a direction: momentum has thus three components, momentum in the x, y and z directions. That means that for the specification of the state of a particle we have three space coordinates plus the time and the three components of the momentum. In quantum mechanics, when you know precisely the momentum of a particle no information on its location can be given. Heisenberg's famous uncertainty relation forbids this. This relation states that the product of uncertainties in position and momentum is larger than some definite number, thus a smaller uncertainty in one implies larger in the other. There is an analogous relation involving time and energy. Thus knowing the momentum (and thereby the energy) precisely there is nothing more to be known. That's it. If you try to find out where the particle is located at what time, you can with equal probability find it anywhere, anytime in the universe. This is quantum mechanics: to compute the probability of a particle to be somewhere one must use waves; to a particle with a definite momentum corresponds a plane wave, one which extends uniformly over all space. A plane wave is like the waves you see on a relatively quiet sea, extending to the horizon and beyond. This is a very strange subject. It is a difficult subject, because it is a situation very different from daily experience. It is easy to "explain" something that everybody can actually see in the macroscopic world that we live in, but particles do not necessarily behave in that fashion. We must treasure those properties that are the same at the quantum level as well as macroscopically. The laws of conservation of energy and momentum belong to those properties. So this is our way of treating the difficulties of quantum mechanics: talk about things you know and understand, and just do not discuss whatever you cannot know. If you cannot know where the particle is located let us not talk about it.

[a]Reminder: at speeds well below the speed of light momentum is simply the product of mass and velocity.

Of course, momentum is never known totally precisely, and one will normally know the location of a particle only in a rough way. With a particle accelerator the particle is part of a beam, and that beam is extracted at some time. So the particle is localized to some extent. But on the microscopic scale these are very, very rough statements, and to deal with a particle exclusively using its energy and momentum is an idealization that for our purposes is close enough to reality. To a particle the beam is the whole universe, and it is big! Here is the scale of things: at a modern accelerator particles are accelerated to, say, 100 GeV, and allowing an uncertainty of 1% in the energy means that as far as quantum mechanics is concerned you can localize it to within one-tenth of the size of a nucleus. An atom is roughly 100,000 times the size of a nucleus, and 100 million atoms make a cm.

So this is what this Chapter is all about: energy and momentum, conservation laws and the mass-shell relation. The discussion will focus on collision processes, the collision of two or more particles. The final state may consist of the same particles but with different momenta and moving in different directions; such processes are called elastic collisions. But it may also happen that the particles in the initial state disappear and other particles appear in the final state. Those are inelastic processes. The conservation laws hold equally well for elastic and inelastic processes, but for inelastic processes there is a difference. The initial particles disappear, and new particles (of which some may be like the initial ones) appear in the final state. Because these particles may have different masses that means that the sum of the initial masses may be different from the total final mass. According to Einstein mass is energy ($E = mc^2$) and therefore this difference in mass implies a difference of energy. That must be taken into account when making up the energy balance. But let us not move ahead of the subject but go about it systematically. Let us state here clearly, to avoid confusion, that when we speak of the mass of a particle we always mean the mass measured when the particle is at rest, not the apparent mass when it moves at high energy.

4.2 Conservation Laws

When thinking about particles most people think of them as small bullet-like objects moving through space. A bit like billiard or snooker balls. There is actually quite a difference between billiard balls and snooker balls: billiard balls are much heavier and do not so much roll as glide over the billiard table while spinning. Billiard balls can be given a spin, which can make their movements quite complicated. As particles generally have spin they resemble billiard balls more than snooker balls. There is another complication in collisions of particles: the particles present after the collision may be very different from those seen initially. A collision of two protons, at sufficiently high energy, may produce a host of other particles, both lighter and heavier than protons. As mentioned before, a collision process where in the final state the same particles occur as initially is called an elastic collision. Billiard ball collisions are elastic collisions, at least if the balls do not break up!

Much of the above picture is correct, although one might do well to think of particles more like blurred objects. Quantum mechanics does that. When particles collide certain conservation laws hold, and some of these laws, valid for particle collisions, also hold for collisions among macroscopic objects as they are made up of those particles. Therefore some of these laws are well-known to us, simply because they can be seen at work in daily life. The foremost conserved quantity is energy: in any collision process the total energy before the collision is equal to the total energy afterwards. Further there is the law of conservation of momentum. There are other conserved quantities (like for example electric charge), but these need not to be discussed in this Chapter. It is good to realize that there may be conservation laws for certain properties that are not at all known on the macroscopic level. One discovers such laws by looking at many, many collision processes and trying to discover some systematics.

Momentum is, for any particle at low speeds, defined as velocity times the mass of that particle. At higher speeds the relationship is more complicated such that the momentum becomes infinite if the velocity approaches the speed of light. Clearly, velocity is not conserved in any process: if you shoot a small object, for example a pea, against a billiard ball at rest then the billiard ball will after the collision move very slowly compared to the pea before the collision. It will however move in the same direction as the pea before the collision. Whatever energy the pea transfers to the billiard ball will have relatively little effect, as that ball is much heavier than the pea. Thus if we are looking for a conservation law the mass must be taken into account, and this is the reason why one considers the product of mass and velocity, i.e. the momentum. So, at the end the pea will be smeared all over the billiard ball, and that ball will have a speed that is the speed of the pea scaled down by the mass ratio, but in the same direction.

Here is a most important observation. **For any object, in particular a particle, momentum and energy are not independent. If the momentum of a particle of given mass is known then its velocity and thus its energy are also known**. This is really the key point of this Chapter. In the following the relationship between momentum and energy will be considered in some more detail. Furthermore, the theory of relativity allows the existence of particles of zero mass but arbitrary energy, and that must be understood, as photons (and perhaps neutrinos) are such massless particles. Also for massless particles the above statement remains true: if the momentum of a particle is known then its energy is known.

Some readers may remember this from their school days: if the velocity of a particle is v then the momentum (called p) of the particle is mv. The kinetic energy is $\frac{1}{2}mv^2$, which in terms of the momentum becomes $\frac{1}{2}p^2/m$. This relation becomes different if the speed is close to the speed of light; relativistic effects start playing a significant role.[b] For a massless particle, always moving at the speed

[b]The precise relation valid for any speed and including the mass energy mc^2 is $E = \sqrt{p^2c^2 + m^2c^4}$.

of light, the energy equals pc, the momentum times c, the speed of light.

It might be noted that velocity has a direction and therefore three components (velocity in the x, y and z direction). Similarly momentum has three components. One says that velocity and momentum are vectors. The relationship between velocity and momentum is vectorial. This means that the relation holds for all components separately, for example the momentum of a particle in the x direction is simply the velocity in the x direction multiplied by the mass of the particle. The conservation of momentum holds for all three components separately, so we have three conservation laws here. The law of conservation of momentum is a law of conservation of a three-dimensional vector. We speak of one conservation law although there are really three separate conservation laws.

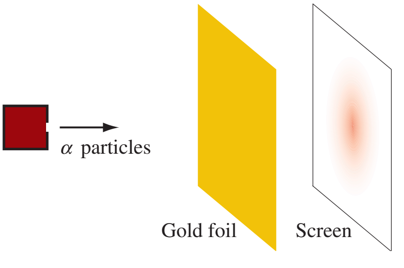
Non-relativistically, the total energy (not including the mass energy mc^2) is the sum of the kinetic energies for each of the components of the velocity, thus the total energy equals $\frac{1}{2}mv_x^2$ plus $\frac{1}{2}mv_y^2$ plus $\frac{1}{2}mv_z^2$.

Consider some elastic collision process of two particles called A and B. Think of something like an electron scattering off a proton, or rather from the electric field of the proton. Let us assume that initially one of the particles (B) is at rest while the other (A) moves in with a certain speed, to exit finally at an angle φ (see figure).

In the figure particle B is supposedly much heavier than particle A so that it barely moves after the impact. What precisely the outgoing angle will be depends on the details of the collision. For billiard balls that depends on where precisely the balls hit each other. For elementary particle collisions one never knows positions in any detail, let alone where the particles hit each other.

Accelerators produce beams of a particular type of particle, and such a beam has a size very, very much larger than that of the object that it is aimed at (the nuclei in the target material). Thus one observes many collisions, and a spectrum of angles. Some particles will come out at small angles, some at larger angles etc. How many come out at a given angle depends on the details of the collision, and on the precise way in which the particles interact. In particle physics one thus studies the angular distribution and tries to deduce properties of the interaction. The angular distribution is the distribution of the secondary particles over the directions. For a given time of beam exposure so many particles exit at 10 degrees (for example), so many at 20 degrees, etc.

Such an angular distribution measurement made its entry into physics at Manchester (England), through the historic experiments of Rutherford and his collaborators. A radioactive source emitting alpha particles (these are helium nuclei, containing two protons and two neutrons) was placed in a box with a small hole. The alpha articles going through that hole would pass through a thin foil of gold and subsequently hit a screen. On the screen a picture would evolve, very intense in the centre and tapering off away from that center. See figure.



This description and the figure do not do justice to the original experiment: many screens completely surrounding the gold foil were used. Anyway, an angular distribution could be deduced. What was stunning to Rutherford was that some of the α particles actually bounced backwards. That could happen because the nucleus of gold is almost 50 times heavier than an alpha particle. The situation is comparable to a ping-pong ball bouncing back from a billiard ball.

Rutherford himself described his reaction to the back-scattering effect as follows: "It was quite the most incredible event that has ever happened to me in my life. It was almost as incredible as if you fired a 15-inch shell at a piece of tissue paper and it came back and hit you."

At that time one knew Coulomb's law concerning the electric force that two charged particles at some distance exert on each other. This law states that the force becomes much weaker if the distance is larger.

Precisely, Coulomb's law states that the force is inversely proportional to the distance squared, so if the particles attract (or repulse) each other with a given force at some distance, then the force will be four times weaker if the distance is twice as large. And the force will be four times as large if the distance is halved.

Consider now a heavy, charged particle as target and scatter a much lighter charged particle off it. The deflection strongly depends on the distance at which the light particle is passing the target. For large distance the deviation will be small, but if the particle passes closely by the target it will be deflected strongly. In Rutherford's experiment the alpha particles were not very well collimated and they were evenly distributed over some area much larger than the size of an atom. The angular distribution will reflect the strength of the force depending on the distance.

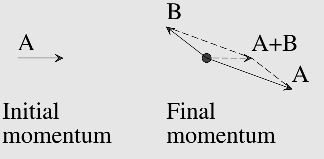
In 1911 Rutherford, shooting alpha particles at a thin metal foil (for example a gold foil), succeeded in deducing Coulomb's law for the interaction between alpha particles and nuclei (both are electrically charged) from the angular distribution of the scattered alpha particles. To an alpha particle, about 7500 times heavier than an electron, the electrons inside the metal foil are of no importance and it "sees" only the nuclei. On the other hand, in the case of gold, the nucleus is about 50 times heavier than an alpha particle, and it barely recoils under its impact.

This experiment made it clear that atoms are largely empty, with a (heavy) nucleus in the centre. Most of the alpha particles did not seem to collide with anything at all, but some of those that did change direction came out at quite large angles, which is what one expects to happen only if the target is much heavier than the projectile. If a billiard ball hits a light object such as a ping-pong ball it will not deflect substantially, while a ping-pong ball will deflect very much if it hits a billiard ball. Rutherford concluded that the nucleus was very heavy and furthermore that it was at least a hundred thousand times smaller than the atom. It was one of the most important experiments of this era; it opened the door for Bohr's model of the atom, formulated in 1913, after Bohr had spent time at Rutherford's laboratory.

Consider again our example, collision of particle A with particle B at rest. Knowing the mass of particle A the momentum of the initial state can be computed from the initial velocity of that particle (the momentum of particle B is zero, as it is at rest). Let us assume particle A comes out at the angle φ with some particular velocity. Then we can compute the momentum of particle A in the final state. Conservation of momentum will allow us then to deduce the momentum of particle B in the final state: its momentum must be such that combined with the momentum of particle A we get precisely the initial total momentum. Thus if we specify the speed and angle of particle A as it exits, we can compute where B goes from the law of conservation of momentum.

However, there is a complication. We can compute the total energy of the initial state. Since the particles in the final state are the same as those in the initial state we need not to take into account the energy implied by their masses, because that is the same finally as initially. Thus, ignoring the mass-energy (the energy associated with the masses of the particles at rest), the initial energy is just the kinetic energy of particle A. That energy must be equal to the sum of the kinetic energies of the secondary particles. That will generally not be the case for the configuration that we discussed; only for a very specific velocity of the outgoing particle A (at that given angle) will the momentum of particle B be such that the energies of particles A and B add up to precisely the initial energy. Thus conservation of energy has as consequence that in a given direction only one specific momentum is possible.

The figure below shows a configuration with conservation of momentum but without energy conservation. The arrows shown depict the momenta of the particles. Outgoing particles A and B have large momenta pointing roughly in opposite directions. The combination of these two is smaller in magnitude and equal to the initial momentum (the combination of the two momenta is the addition of vectors: one must draw a parallelogram).



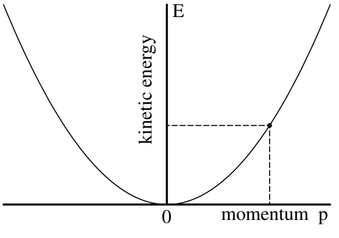
Since the magnitudes of the momenta of A and B are clearly larger than that of A initially (this is depicted by the length of the arrows in the figure), the energies of the final A and B are larger than that of the initial A. Obviously, energy conservation is violated, as the energies of both particles and therefore also their sum exceeds the energy of the initial state.

Briefly, the fact that energy and momentum of a particle are not independent has as consequence that for a given direction for the exiting particle A there is only one specific momentum allowed for that particle, in order for energy to be conserved. Particle A may still exit in all possible directions, but for a given direction the momentum (the speed) is fixed by the law of conservation of energy.

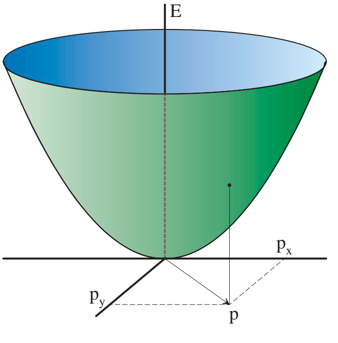
In Rutherford's experiment the mass of the target particle is much larger than that of the incident alpha particle, and in such a case the target particle moves only very slowly after the collision. The energy absorbed by the target is then negligible, and therefore the energy, and consequently the velocity of the outgoing alpha particle is practically the same as before the collision. It just bounces off the nucleus. The alpha particle has four nucleons (two neutrons and two protons), and the target materials used by Rutherford were gold (whose nucleus contains 197 nucleons) and aluminum (27 nucleons). The alpha particle was really the ideal projectile for this experiment: not too light (much heavier than the electrons in the atoms) and not too heavy.

The relation between momentum and energy for a particle of a given mass is a very important relation that will play a central role later on. This relation has a name: it is called the mass-shell relation. It is called that way because of the mathematical figure that one may associate with this relation. Let us make a plot of the energy of a particle versus its momentum.

Momentum is normally a three-dimensional vector, but for the moment we restrict ourselves to a momentum in only one direction. Then we can make a plot, with that single component of momentum along the horizontal axis. We must allow positive and negative values (movement of the particle to the right or the left respectively). We then have for the associated energy a parabola. Remember, the relationship is quadratic: if the energy has some value for a given momentum, then it will be four times greater if the momentum becomes twice as large.



Relationship between momentum and energy of a particle of given mass. Chose some momentum, draw a vertical line (dashed) until it hits the curve and then draw a horizontal line to the energy axis. The energy there corresponds to the momentum chosen



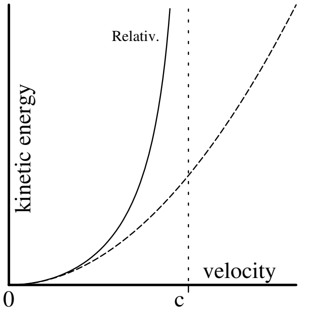
The same for the case of a two-component momentum

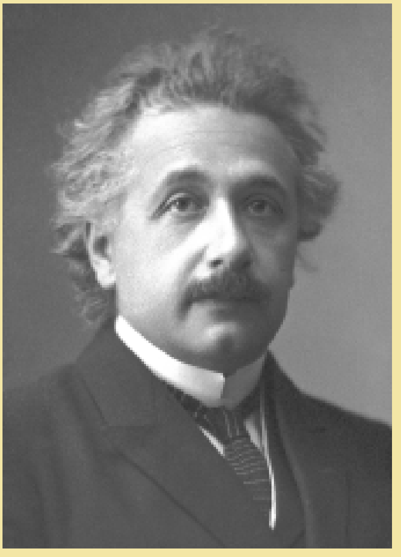
If we want to make a plot for the case where the momentum has two components we get a 3-dimensional figure obtained from the previous figure by a rotation around the energy axis. The momentum in the x direction is plotted along the horizontal axis in the plane of the paper, the momentum in the y direction along the axis perpendicular to the paper. That figure looks like a shell, and physicists call it the mass-shell. For a given momentum p, with components p_x and p_y the corresponding energy can be found as shown in the figure. Given p_x and p_y construct the point p. Then draw a line straight upwards from that point p. It will intersect with the shell. The length of the line from p up to the intersection with the shell is the energy E associated with the momentum p.

For relativistic particles, i.e. particles moving with speeds that approach that of light the curve differs slightly from a parabola, as will be discussed in the next section.

4.3 Relativity

If particles have velocities approaching the speed of light relativistic effects become important. The kinetic energy and the momentum depend on the velocity in a different way, namely such that for speeds approaching the speed of light (c) both energy and momentum go to infinity. The speed of light can never be reached, as the energy needed is infinite. That is the way the limit of the speed of light is imposed by the theory of relativity.





Albert Einstein (1879–1955). The magic year, 1905, when Einstein produced four revolutionary papers (photon, theory of relativity, $E = mc^2$, and an explanation of Brownian motion) was in the period 1902–1908 that he worked at the patent office in Bern. He was actually quite happy there, he liked the work and received reasonable pay. Also his superior was quite happy with him: he was called one of the most esteemed experts at the office. The great advantage of this job was that it left him enough time to do his physics research.

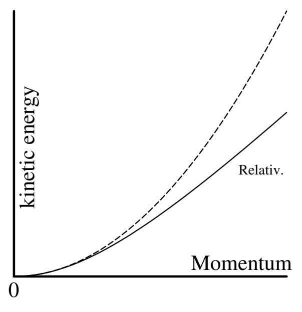
Here are two Einstein anecdotes, of which there are remarkably few.

At some occasion Einstein was received, together with Ehrenfest, by the Dutch queen. As Einstein did not have any formal suit he borrowed one from Ehrenfest; in turn Ehrenfest dug out from his wardrobe some costume that emitted a strong moth-ball odour. This did not go unnoticed by the royalty. As Einstein remarked afterwards: “The royal nose was however not capable of determining which of us two was stinking so badly.”

When asked: “What is your nationality?”, Einstein answered: “That will be decided only after my death. If my theories are borne out by experiment, the Germans will say that I was a German and the French will say that I was a Jew. If they are not confirmed, the Germans will say that I was a Jew and the French will say that I was a German.” In actual fact, Einstein kept his Swiss nationality until his death, in addition to his US citizenship.

In the figure the dashed curve shows the energy versus the velocity in the pre-relativistic theory, the solid curve shows the same relationship in today's theory. In experimental particle physics one practically always deals with ultra-relativistic particles, with speeds within a fraction of a percent (such as 1/100%) from the speed of light. It is clearly better to work directly with momentum rather than with velocity.

The relation between energy and momentum changes much less dramatically when passing from the pre-relativistic formulation to the relativistic theory. In fact, the energy increases less sharply with momentum, and for very high values of the momentum the energy becomes proportional to it. (Energy approximately equals momentum times c, the speed of light.) A typical case is shown in the next figure, with the dashed line showing the non-relativistic case, the solid curve the relativistically correct relation.



The quantitatively minded reader may be reminded of the equations quoted in Chapter 1. In particular there is the relation between energy and momentum, plotted in the next figure:

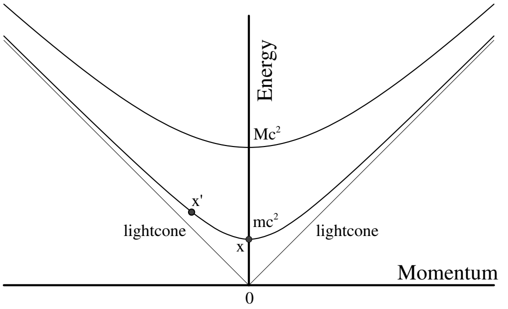
$$E = c\sqrt{p^2 + m^2c^2}\,.$$

or, using the choice of units such that $c = 1$:

$$E^2 = p^2 + m^2\,.$$

Another important fact is the Einstein equation $E = mc^2$. This very famous equation can be deduced in a number of ways, none of which is intuitively appealing. This equation tells us that even for a particle at rest the energy is not zero, but equal to its mass multiplied with the square of the speed of light. In particle physics this equation is a fact of daily life, because in inelastic processes, where the set of secondary particles is different from the primary one, there is no energy conservation unless one includes these rest-mass energies in the calculation. As the final particles have generally masses different from the primary ones, the mass-energy of the initial state is in general different from that of the final state. In fact, the first example that has already been discussed extensively is neutron decay; this decay is a beautiful and direct demonstration of Einstein's law, $E = mc^2$. Indeed it is in particle physics that some very remarkable aspects of the theory of relativity are most clearly demonstrated, not just the energy-mass equation. Another example is the lifetime of unstable particles, in particular the muon. The lifetime of a moving muon appears to be longer in the laboratory, in accordance with the time dilatation predicted by the theory of relativity.

Thus the mass-energy must be included when considering the relation between energy and momentum. The figure shows the relation between energy and momentum for two different particles, respectively with masses m and M. We have taken M three times as large as m. For zero momentum the energy is simply mc^2 for the particle of mass m and Mc^2 for the particle of mass M.



This figure is really the all-important thing in this Chapter. Understanding it well is quite essential, since we shall draw a number of conclusions from it. In itself it is simple: the curve shows the relation between momentum and energy for a single particle. Given the momentum of a particle of mass m one can find the corresponding energy by using the curve for mass m. If the momentum is zero then the energy is mc^2.

In drawing the figure one must make a choice of units. We have drawn a figure corresponding to a choice of units such that the speed of light is one. For very large positive or negative momenta energy becomes very nearly equal to the magnitude of the momentum. In the figure that we have drawn the diagonal lines represent the relation energy = ± momentum. The curves approach these diagonal lines for large momenta. The diagonal lines define the light cone; the reason for that name shall become obvious soon.

To draw the figure we assumed the momentum to have only one non-zero component; if the momentum is in a plane (has two non-zero components) the figure becomes three-dimensional, and can be obtained by rotating the figure shown here around the energy axis. We then have two mass-shells and one cone, the light cone.

One of the results of the theory of relativity is that the velocity of a particle equals the ratio of its momentum and energy (in units where the speed of light is one). So for any point on any one of the curves the velocity is the ratio of the horizontal and vertical coordinates of that point. For large momenta the ratio becomes one (the curve approaches the diagonal line) and the particle moves with a speed very close to one, the speed of light.

In Chapter 1 we gave the relation between momentum, energy and velocity, in particular

$$v = \frac{p}{E}.$$

Here units such that $c = 1$ were assumed. If the particle moves slowly the energy of the particle is very nearly equal to its rest energy, i.e. to $mc^2 = m$. Then $v = p/m$, or $p = mv$.

An interesting point that can be seen from the figure is what happens if we consider the zero mass limit. So, imagine the curve that you get if the point x (fat dot, with mc^2 written beside) is pulled down, to zero. Then obviously the curve becomes the light cone. Thus zero mass particles are perfectly possible, and their energy is equal to the magnitude of the momentum. They always move with the speed of light as the ratio of momentum and energy is always one for these lines. The photon is such a zero mass particle. It has a well defined energy and momentum. Other particles of zero mass are the neutrinos (although there is some question whether their masses are really zero or just very small). Particles definitely of zero mass are the gluons, the basic constituents of the strong forces, and the graviton, responsible for the forces of gravitation.

Finally, the figure may also serve to see what happens to energy and momentum of a particle when changing the reference system from which the particle is observed. First, consider a particle of mass m at rest. The energy will be mc^2, the momentum zero. This is the point x. Now go to a system moving with some velocity v with respect to the particle. In that system the velocity of the particle will be $-v$. The momentum will be what you get by multiplying $-v$ by the mass of the particle. The energy can likewise be computed from this velocity; the momentum and energy are of course related as given by the curve that we have plotted. Thus the new point x' corresponding to the values of energy and momentum in this moving system will be somewhere on the same mass-shell, for example as indicated in the figure. Stated differently: it is impossible to say whether we (the reference system) move or if we are at rest and the particle moves. The relation between momentum and energy is the same. That is in fact precisely the idea of relativity.

4.4 Relativistic Invariance

It was Einstein's theory of relativity that emphasized and made explicit the important role of invariance principles. Already since

Newton and Galilei a number of important laws were generally accepted. For example, there is the idea of rotational invariance: physics in two reference systems that differ from each other by their orientation is the same. Let us formulate this slightly differently. Imagine that two physicists are deducing laws of physics by doing experiments, each in his own laboratory. However, the two laboratories are not quite identical: they are oriented differently, although otherwise there is no difference. For example, imagine that one does his experiments during the day, and when he leaves somebody rotates his whole laboratory over a certain angle, after which the second physicist does his work at night. In the morning the laboratory is rotated back etc. Invariance under rotations means that these two physicists arrive at precisely the same conclusions, the same fundamental laws, the same constants. They measure the same spectral lines when heating up gases, deduce the same laws of electricity (Maxwell's laws), arrive at the same laws of motion etc. Of course, if each of them were to look to the other they would see that they are differently oriented, but it is easy to transform configurations into each other once you know the angles of rotation.

Most people accept this kind of invariance as self-evident. Other examples are translational invariance in both space and time: laws of physics deduced in Europe are the same as those seen in the US, or on the moon. And we also think that the laws of physics are the same today as yesterday or tomorrow. While indeed these invariances do not particularly surprise us, it is only in the twentieth century that we have come to understand their importance. Much of that is due to the fact that things have become much less self-evident with the introduction of the theory of relativity, forcing us to scrutinize these principles more closely. Einstein's theory of relativity was explicitly built upon two principles (in addition to rotational and translational invariance):

— Equivalence of reference systems that are in motion (with constant speed) with respect to each other;

— The speed of light is the same when measured in systems in motion (with constant speed) with respect to each other.

The first principle was already part of physics well before Einstein; it is the second statement that causes effects that are not self-evident. If light is emitted from a moving object one would not expect that light to move with the same speed as a ray coming from an object at rest. You would expect a difference equal to the speed of the moving object. Imagine someone throwing a stone forward while being on a moving train. We would expect that someone standing beside the train would see this stone coming at him with a speed that is the sum of the speed of the stone (as seen on the train) and the speed of the train. Even if the person in the train would merely just drop the stone, the other outside would see that stone coming to him with the speed of the train. Thus the speeds measured on the train or outside the train are not the same. Yet the theory of relativity says that if the stone moves with the speed of light on the train, also the person outside will see it moving with that same speed (we leave aside that it requires infinite energy to get a stone to move with the speed of light). What happened to the speed of the train? Something is funny here. Einstein shifted the problem: a speed measurement implies measurement of distance and time, and these are different from what we normally think, and depend on the state of movement. Thus, there is something funny with time and space measurements. The relationship between a measurement of distance and time of some event by a person on the train to a measurement of the same event by a person outside is very strange to us. In other words, if the person on the train measures the speed of a stone thrown from the train then the speed of that stone measured by the person outside is **not** what you would think, namely the velocity measured on the train plus the speed of the train. To be sure, the deviation is small unless the velocity is in the neighbourhood of the speed of light, so in daily life we see nothing of these effects. The strange thing is the constancy of the speed of light, and that causes all these consequences with respect to measurements

Papers that changed the world: E = mc2.

Annalen der Physik 20 (1905) 639

13. ***Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?***
von A. Einstein.

Die Resultate einer jüngst in diesen Annalen von mir

Die Masse eines Körpers ist ein Maß für dessen Energieinhalt; ändert sich die Energie um L, so ändert sich die Masse in demselben Sinne um $L/9.10^{20}$, wenn die Energie in Erg und die Masse in Grammen gemessen wird.

Bern, September 1905.

In this short (3 pages) paper Einstein presents a derivation of the relation $E = mc^2$. He explicitly gives the equation in words, in the form $m = E/c^2$. The square of the speed of light is given as 9×10^{20}, indeed the square of $c = 3 \times 10^{10}$ cm/s. Energy is denoted by L.

It is interesting to note that he does not present this equation saying how much energy is contained in a given amount of mass. Instead he says: you can measure the energy of a body by measuring its mass. He did not think of mass of a body as a source of energy, rather he saw it as a way of measuring the energy contained in that body. Whether you can get it out is an entirely different matter.

Is the Inertia of a Body Dependent on its Energy Content?

by A. Einstein

The results of a recently published investigation by me in these Annals...

The mass of a body is a measure of its energy content: if the energy changes by an amount L then the mass changes in the same sense by $L/9 \times 10^{20}$ if the energy is given in ergs and the mass in grams.

of space and time. Let us give an example of the uncanny effects that occur.

The most direct way in which particle physicists meet the effects of relativity is when measuring the lifetime of an unstable particle. Muons, copiously produced by cosmic rays and also at particle accelerators, fall apart after a rather short time (in about two millionths of a second). However, measuring the lifetime for a slow moving muon or a muon moving with high speed gives different results: the fast moving muon lives longer. It is a direct manifestation of the effects of relativity, and a fact of daily life at the particle accelerators. When the lifetime of a certain particle is reported one must specify its state of motion. In the tables used by particle physicists the lifetime is usually understood to be the time measured when the particle is at rest. There is a similar effect when measuring distances. The precise equation relating distance measurements was deduced by Lorentz even before Einstein introduced the theory of relativity; this is the reason why one speaks of a Lorentz transformation when relating quantities measured in reference systems moving with respect to each other.

Finally, from the discussion before, we know that a point on the mass-shell (corresponding to a particle of definite mass, momentum and energy) will transform under a Lorentz transformation to another point on the mass-shell. Precisely how x became x' as discussed above. The Lorentz transformation specifies precisely where the point x' will be, given x and the relative velocity of the systems. In this sense the mass-shell is an invariant: a particle of given mass remains on the same mass-shell when going to another reference system. For a particle of zero mass we have the light cone, and going to a differently moving system a point on the cone (for which energy equals the magnitude of the momentum) will become another point on the cone (where again energy equals the magnitude of the momentum). The values of energy and momentum however will of course be different.

We have thus a number of invariances in physics. The invariance of the laws of physics with respect to rotations and with respect to systems moving with a constant speed relative to each other is now generally called Lorentz invariance. Including invariance with respect to translations in space and time one speaks of Poincaré invariance. Both Lorentz and Poincaré made their contributions prior to Einstein; it is Einstein who invented relativistic kinematics and made us understand the whole situation in full clarity.

Invariance with respect to relative movement can be used with advantage to understand certain situations. If some physical process is forbidden (or allowed) in some system it is forbidden (or allowed) in systems that move relative to that original system. For example, if some decay process does not occur if a particle is at rest it will also never occur if it moves; we shall effectively use this seemingly trivial observation to clarify complex situations. Deducing things in the most convenient reference frame is often of great help in particle theory.

4.5 The Relation $E = mc^2$

The equation $E = mc^2$ is surrounded by mystique, and there is the folklore that this equation is somehow the starting point for making an atomic bomb. It might not do any harm to explain this equation in some detail, to demystify it.

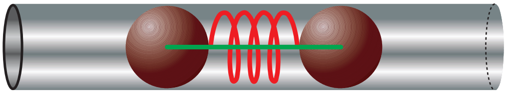
In the simplest possible terms this equation means that energy has mass. Given that the weight of an object is proportional to its mass this means that energy has weight. Consider an old-fashioned watch, with a spring that must be wound regularly. When the spring is completely unwound, measure the weight of the watch. Then wind it, meaning that you put energy into the spring. The energy residing in the spring has some weight. Thus if you measure the weight of the watch after winding the spring it will be a little heavier. That weight difference is very small but non-zero.[c] You need really a massive amount of energy before the additional

[c]It is something like one hundred-millionth-millionth part of a gram.

weight becomes noticeable. A little bit of mass corresponds to a very large amount of energy. That is because the speed of light is so large. A radio signal goes seven times around the earth in one second.

Here another example. Take a car, weighing, say, 1000 kg. Bring it to a speed of 100 km/h. The weight of the corresponding energy is one half of the hundred-millionth part of a gram $(0.5 \times 10^{-8}\ \text{g})$.[d] You can see that energy weighs very little; no wonder that nobody ever observed this effect before Einstein came up with his famous equation. It took a while (till about 1937) before it was demonstrated explicitly.

As yet another example consider a double sided cannon. This is a type of cannon that might be useful if you are surrounded, and that fires two cannonballs in opposite directions. Thus there is a long cannon barrel, and one inserts a cannonball at each end. You could imagine gunpowder between the two balls, but here we will suppose that there is a very strong spring that is pushed together. Once pushed as far as possible a rope is attached that keeps the two balls together. At the command "fire" some person cuts the rope and the two balls will fly off in opposite directions, with a velocity determined by the amount of energy stored in the spring.



The above figure shows this idea, the green line is the rope keeping the balls together. Measure very carefully the weight of the cannon barrel, the two cannonballs, the rope and the spring

[d] It can be computed by evaluating $\frac{1}{2}Mv^2/c^2$ where M is the weight of the car at rest in grams while v is the speed of the car, about $\frac{1}{36}$ km/s, and c the speed of light, 300 000 km/s.

before pushing in the balls. Then push in the two balls. That will cost you energy, and that energy will be stored in the spring. Next make again a measurement of the weight of the whole ensemble. The result will be that the total is now slightly heavier than with uncompressed spring.

A decaying neutron has much in common with our double sided cannon. To paraphrase Einstein, God throws his dice, and when a six comes up he cuts the rope. Thus when the neutron decays, two particles, an electron and a neutrino, shoot away (not necessarily in opposite directions), and a proton remains more or less at the place of the neutron. Here the energy is relatively large: the difference between the neutron mass and the sum of the proton and electron mass (the neutrino mass is very small or zero) is about 0.1% of the neutron mass. It translates into kinetic energy of the electron and the neutrino, the proton remains practically at rest. In particle physics Einstein's equation is very much evident in almost any reaction.

Now what about the atomic bomb? The function of the equation $E = mc^2$ is mainly that one can tell how much energy becomes available by simply weighing the various objects taking part in the process. A uranium nucleus becomes unstable when a neutron is fired into it, and it breaks up in a number of pieces (including several neutrons, which can give a chain reaction). The pieces are nuclei of lighter elements, for example iron. Since the mass of the uranium nucleus is well-known, and since the masses of the various secondary products are known as well, one can simply make up the balance (in terms of mass). The difference will be emitted in the form of kinetic energy of the decay products, and it is quite substantial. So that is what Einstein's equation does for you: you can use it to determine the energy coming free given the weight of all participants in the process. Perhaps it should be added that in the end the kinetic energy of the decay products will mainly translate into heat (which is in fact also a form of kinetic energy of the molecules). The real energy producing mechanism here resides in the way the protons and neutrons are bound together in the nuclei.

During his life Einstein used different methods to derive his equation. Originally he took the hypothetical case of an atom at rest emitting light in opposite directions, so that the momentum of the atom was zero both before and after the emission. Then he considered how this looks from a system moving with respect to this radiating atom. He knew precisely how the light rays looked in the moving system: for that one uses the light cone. Assuming conservation of energy he could state quite precisely what the energy difference was between the initial and final state of the atom. Thus he looked at it in two different systems: one in which the atom is at rest both before and after the emission, and one where the atom had momentum both before and after. He also knew precisely the difference in energy of the atom between the two cases, because that was equal to the difference of the energy of the light if the system is at rest and the energy of the light in the moving system. In other words, he got a piece of the curve, and from there his equation follows. To say it slightly differently: once he knew about the light cone, he could deduce what happened in other cases by considering what happens if light is emitted. Conservation of energy is the key to his derivation in all cases.