UCD Physics 9A Lab Manual

The Staff of the UCD Physics Dep't
Rev. 9/1998

DIFFERENTIAL APPROXIMATIONS

In dealing with small differences such as changes or uncertainties in measured quantities *x*, we often wish to know how large a difference this will cause in a calculated result *f(x)*. A simple approximation for the relation between these differences $\Delta f \equiv f(x) - f(x_0)$ and $\Delta x \equiv x - x_0$ is therefore very useful, and a rough understanding of the nature of the approximation, and of higher order terms in the series for the exact difference is helpful.

For most purposes, the ratio of the differences is extremely well approximated by the derivative,

$$\frac{f(x) - f(x_0)}{x - x_0} \equiv \frac{\Delta f}{\Delta x} \quad \frac{df(x_0)}{dx_0} \equiv f'(x_0) \qquad (1)$$

This is often called a <u>differential approximation</u> since it become an exact relation (the definition of the derivative) in the limit of the differences becoming differentials, that is, infinitesimally small. This is valid in discussions of uncertainties since uncertainties are presumed to be quite small and, moreover, only need to be known approximately, even for extremely accurate results (for example, *f* may be measured to an accuracy of eight parts per million while the eight parts accuracy might only be determined to be between seven parts and nine parts.)

For a proof and further understanding of the differential approximation (1) we may use the Taylor's series (see Eq. (1) in our section "Series Expansions") in the form

$$\frac{\Delta f}{\Delta x} = f'(x_0) + f''(x_0)\frac{\Delta x}{2} + f'''(x_0)\frac{(\Delta x)^2}{6} + \quad . \qquad (2)$$

This not only becomes the differential approximation in the limit of small Δx, it also provides an exact expansion for Δf that can be used for other purposes. For example, if we happen to have $f'(x_0) = 0$, then the differential approximation $\frac{\Delta f}{\Delta x} \quad 0$ may be too simple and can be improved to $\frac{\Delta f}{\Delta x} \quad f''(x_0)\frac{\Delta x}{2}$. This also clarifies the relative meaning of "small", using the terminology "n-th order" for an effect proportional to $(\Delta x)^n$. Thus the second order term $f''(x_0)(\Delta x)^2 / 2$ is small compared to the first-order differential approximation $\Delta f \quad f'(x_0)\Delta x$ unless the first-order term is zero ($f'(x_0) = 0$). The importance of this distinction may be seen by considering the effect of an "order-of-magnitude" improvement in a measurement *x* (that is, improvement sufficient to reduce its uncertainty by a factor of 10). If this has a first-order effect on a calculated quantity *f*, then the uncertainty of that quantity *f* is improved by an order of magnitude, say from $\pm 20\%$ to $\pm 2\%$; but if this were a second-order effect, the improvement would be <u>two</u> orders of magnitude, from $\pm 20\%$ to $\pm 0.2\%$!

For function of <u>two</u> variables, the Taylor's series for the difference $\Delta f \equiv f\big(x, y\big) - f(x_0 y_0\big)$ *is*

$$\Delta f = f_x \Delta x + f_y \Delta y + f_{xx}\frac{(\Delta x)^2}{2} + f_{yy}\frac{(\Delta y)^2}{2} + f_{xy}(\Delta x)(\Delta y) + \qquad (3)$$

where the notation may be deduced by comparison with the geometric examples below or Eq. (4) in the section on Series Expansions.

A <u>geometrical view</u> of these approximations is that any smooth curve *y=f(x)* is well approximated near any point $y_0 = f(x_0)$ by the <u>line</u>

$$y = y_0 + (x - x_0) f'(x_0) \qquad (4)$$

that is <u>tangent</u> to the curve at that point. Near any <u>minimum</u> or maximum point $(f'(x_0) = 0)$ the curve is approximately the <u>parabola</u>

$$y = y_0 + (x - x_0)^2 f''(x_0)/\ 2; \qquad (5)$$

$f''(x_0) > 0$ for a minimum, $f''(x_0) < 0$ for a maximum. A *smooth surface in three dimensions, z= f(x,y), is well approximated near any point* $z_0 = f(x_0, y_0)$ *by the <u>tangent plane</u>*

$$z = z_0 + (x - x_0)\frac{\partial f}{\partial x_0}(x_0, y_0) + (y - y_0)\frac{\partial f}{\partial y_0}(x_0, y_0). \qquad (6)$$

Near a <u>minimum</u> or maximum $\left(\frac{\partial f}{\partial x}=0 \text{ and } \frac{\partial f}{\partial y}=0\right)$ *any surface is approximately the <u>elliptical paraboloid</u>*

$$z = z_0 + \frac{(x - x_0)^2}{2}\frac{\partial^2 f}{\partial x_0^2}(x_0, y_0) + \frac{(y - y_0)^2}{2}\frac{\partial^2 f}{\partial y_0^2}(x_0, y_0) + (x - x_0)(y - y_0)\frac{\partial^2 f}{\partial x_0 \partial y_0}(x_0, y_0); \qquad (7)$$

the z=constant cross sections are ellipses whose major axis is not along the x or y axis unless $\frac{\partial^2 f}{\partial x \partial y}=0.$ *As in (5), (7) is a minimum or maximum depending on whether the sign of* $\frac{\partial^2 f}{\partial x^2}$ *and* $\frac{\partial^2 f}{\partial y^2}$ *is positive or negative; if* $\frac{\partial^2 f}{\partial x^2}$ *and* $\frac{\partial^2 f}{\partial y^2}$ *have opposite signs, the point is neither a maximum nor minimum but is the saddle point in a hyperbolic paraboloid.*