UCD Physics 9A Lab Manual

The Staff of the UCD Physics Dep't

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SERIES EXPANSIONS

Taylor's Theorem states that any function *f(x)* can be expanded about any point x_0 as a power series

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2} + \ldots \qquad (1a)$$

$$= \sum_{n=0}^{\infty} f^{(n)}(x_0)\frac{(x - x_0)^n}{n!} \qquad (1b)$$

assuming that all derivatives $f^{(n)}(x_0) \equiv \left.\frac{d^n f}{dx^n}\right|_{x = x_0}$ exist at x_0 Only the first few terms of the series are needed for good accuracy for any x close to x_0 For better accuracy, we must either move x_0 closer to x or include more terms of the series. Example: To expand $f(x) = e^x$ about $x_0 = 0$ we note $\frac{d^n e^x}{dx^n} = e^x$, so the coefficients $f^{(n)}(x_0) = e_0 = 1$ are the same for all n. Thus the Taylor's series (1) is :

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \ldots$$

(2a)

The results of using only the two leading terms,

$$e^x \quad 1 + x, \qquad (2b)$$

are compared with the exact function in Table 1, where we see that the error is no more than 1% for $x < 0.15$. Including the next term $(x^2/2)$ greatly improves this accuracy to 0.05% at $x = 0.15$ (see Question 1).

Table 1

x	e^x	$1+x$	% error
0	1	1	0%
0.05	1.051	1.050	(-)0.1%
0.1	1.105	1.1	(-)0.5%
0.15	1.16	1.15	(-)1.0%
0.2	1.22	1.20	(-)1.8%

If f is a function of another variable, y, as well as x, then each of the coefficients $f^{(n)}(x_0) = \frac{d^n f}{dx_0^n}(x_0)$ *in the power series (1) is also a function of y and can be expanded in its own power series*[*]

$$f^{(n)}(x_0, y) = f^{(n)}(x_0, y_0) + (y - y_0)\frac{\partial f^{(n)}}{\partial y_0}(x_0, y_0) + \frac{(y - y_0)^2}{2}\frac{\partial^2 f^{(n)}}{\partial y_0^2}(x_0, y_0) + \ldots \qquad (3)$$

Substitution of this y series (3) into the x series (1) yields the combined series

$$f(x, y) = f(x_0, y_0) + (x - x_0)\frac{\partial f}{\partial x_0}(x_0, y_0) + (y - y_0)\frac{\partial f}{\partial y_0}(x_0, y_0)$$

[*] *The partial derivative notation* $\frac{\partial f}{\partial y}(x, y)$ *means that x is held constant for the differentiation with respect to y, similarly* $f^{(n)}(x_0, y) = \frac{\partial^n f}{\partial x_0^n}(x_0, y)$ *means differentiation with respect to* x_0 *while y is held constant. See a calculus text for more complete discussion.*

$$+\frac{(x-x_0)^2}{2}\frac{\partial^2 f}{\partial x_0^2}(x_0,y_0)$$

$$+(x-x_0)(y-y_0)\frac{\partial^2 f}{\partial y_0 \partial x_0}(x_0,y_0)$$

$$+\frac{(y-y_0)^2}{2}\frac{\partial^2 f}{\partial y_0^2}(x_0,y_0)+\ldots \qquad (4)$$

Dependence on additional variables (z,...) could be expanded in similar combined series.

Different notation is given and the geometrical meaning of the leading terms in (1) and (4) is discussed in the section on Differential Approximations.

<u>QUESTIONS</u>

1. Repeat the example using one extra term: $e^x \approx 1+x+\frac{x^2}{2!}$. Now how large can x be before the error is 1%?

2. How many terms are needed in Equation (2A) for 1% accuracy at $x=1$?

3. Small angle approximation:

   (a) Use Taylor's Theorem to prove that for small angles the following are true:

   $$\sin\theta \approx \theta \text{ and } \tan\theta \approx \theta$$

   (Note θ must be in <u>radians</u> not <u>degrees</u>!)

   (b) Find the next non-zero terms in these expansions.

4. Derive the first three terms for the expansion of $\cos\theta$ about $\theta_0 = 0$. Why is $\cos\theta \approx 1$ a reasonable expression?

5. Use Taylor's Theorem to show that $\frac{1}{1+x} = 1 - x + x^2 - x^3 \ldots$

6. Use Taylor's Theorem to show that e^x expanded about $x_0 = 1$ to first order is $e^x \approx 2.72x$.

   Over what values of x is the error in this expression less than 1% ?

7. Use Taylor's Theorem to show the Binomial Theorem:

   $$(x+y)^n = x^n + \frac{n}{1!}x^{n-1}y$$
   $$+\frac{n(n-1)}{2!}x^{n-2}y^2$$
   $$+\ldots \text{ for } y^2 < x^2$$

   (Hint: Expand in terms of y about $y_0 = 0$).

8. Show any function has the shape of a parabola near its minimum.