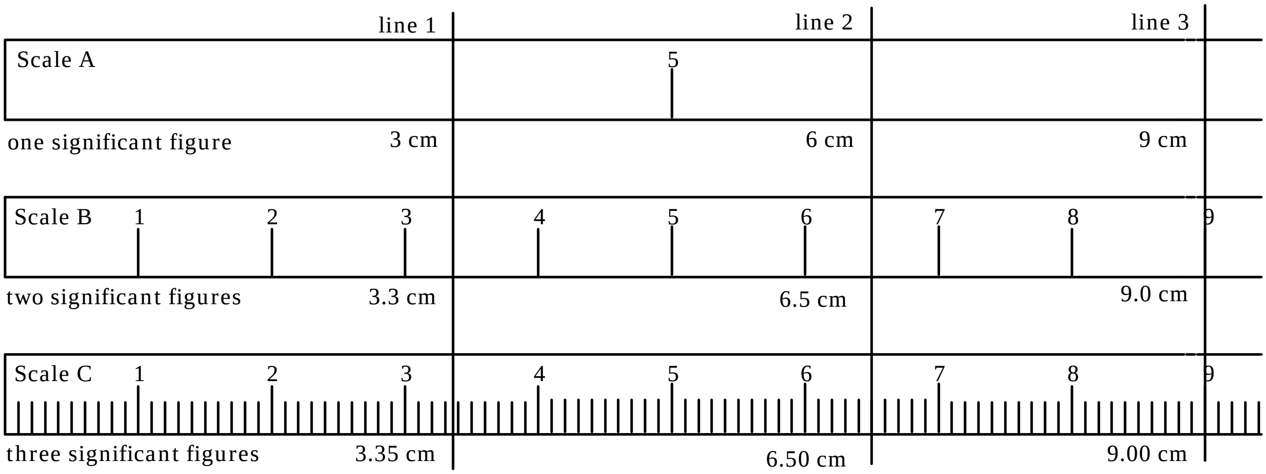
Analysis of Experimental Measurements

Think carefully about the process of making a measurement. A measurement is a comparison between some unknown physical quantity and a standard of that physical quantity. As an example, consider measuring the length of an object with a meter stick. You lay the meter stick next to the object and determine its length by how the ends of the object line up or compare with the marks on the meter stick. The measurement compares the length of the object to the length of a standard meter. The meter stick is a device that aids in making the comparison. Because all measurements are actually comparisons to standards of physical quantities, all measured values are estimates and are therefore not exact values.

Since all measured values are only estimates and not exact values, knowing the estimate by itself is not sufficient. It is quite important to know how good the estimate is, otherwise, the estimate itself is not very meaningful. Consider measuring the height of a doorway. You could make a rough measurement by using your body as a crude instrument if you know your height. A reasonable estimate of the height of the doorway would be 7 feet. Now along comes a carpenter that needs to fit a door into the doorway. Can the carpenter use your estimate to cut a door to fit? Your answer is probably no. You know your answer may be off by several inches and that just wouldn't be close enough for cutting a door to fit. Unless you indicate to the carpenter how your measurement was obtained, the carpenter has no idea how good your value of 7 feet is.

Physics, more than any other science, is based on accurate measurements of primitive quantities, such as mass, length, and time. It is thus appropriate that you gain some understanding about the basic principles of measurements and the treatment of errors or uncertainties that are invariably associated with these measurements.

First we will discuss significant figures. Consider a measurement of the positions of the three lines in the figure below.



If we use scale A then we might measure the positions of the three lines as (1): 3, (2): 6, (3): 9 (in units of cm), while with scale B we might obtain (1): 3.3, (2): 6.5, (3): 9.0. Finally with scale C: (1): 3.35, (2): 6.50, (3): 9.00.

In each case the numbers are the best representations of the measurements with the given scale, limited by the markings on the scale. Significant figures are defined to be the least number of digits which give the best representation of a measurement in accordance with the accuracy limitation of the measuring device. In the example above we were limited to one significant figure when we used scale A while with scale B we had we had two significant figures and with scale C, three significant figures. Notice that with scale B, in the case of line (3), or with scale C in the case of lines (2) and (3) we needed to include zeros as significant digits in places where normal usage might lead to their being dropped. To avoid confusion as to whether a zero is a significant digit a sensible procedure would be to underline a terminal significant zero, e.g. $9.\underline{0}$ or $9.0\underline{0}$ for line (3) with scales B and C respectively. This is especially true in the case of numbers like 9000, where any of the three zeros could be the last significant digit; this problem is somewhat alleviated by the use of scientific notation e.g. $9.0\underline{0} \times 10^3$ would indicate three significant figures.

In combining numbers one must be cognizant of the number of significant figures in each. If we consider the following sum in which doubtful digits (lowest order significant digits) are underlined:

$$\begin{array}{r} 12.\underline{7} \\ 3.57\underline{4} \\ 0.1\underline{6} \\ \hline 16.434 \end{array}$$

If you rounded the numbers $3.57\underline{4}$ & $0.1\underline{6}$ before adding them, you would get $16.\underline{5}$. By adding and then rounding, you get a better value $16.\underline{4}$.

We see that the last three figures are doubtful. There is no point in retaining the last figures in the hundredths or thousandths place since there is already a doubtful figure in the tenths place (the 7 in 12.7 could be 6 or 8). We thus arrive at the following rule; Do not retain any digits in an addition or subtraction beyond the first decimal place which contains a doubtful figure. In the above example we thus can round all numbers to the nearest tenth since the tenths place is already doubtful. For multiplication or division we need a different rule. Consider the product:

$$\begin{array}{r} 13\underline{4} \\ \times\ 1\underline{2} \\ \hline \underline{268} \\ 13\underline{4} \\ \hline 1\underline{608} \end{array} \qquad \begin{array}{l} 1\underline{61}0 \text{ (intermediate result)} \\ 1\underline{6}00 \text{ (final result)} \end{array}$$

We see the uncertain 2 in $1\underline{2}$ makes the last three figures uncertain. The product should thus be written $1\underline{6}00$. Rule: In a multiplication o11r division, the number of figures retained in the answer is the number of significant figures in the number which has the fewest significant digits. Thus in the example, the 12 has only two significant figures while 134 has three, the product has only two.

Note that the above rule pertains to final answers. If the computation is not a final result but only an intermediate result it is reasonable to retain one extra digit in a product or quotient i.e. the above result could be rounded to $1\underline{61}0$ if it were to be used further.

Measuring uncertainties and errors.

The result of a measurement yields a numerical representation in which the number of significant digits is usually determined by the measuring instruments and our ability to read them. The last significant digit on the right (the least significant digit) has an uncertainty or error associated with it which can be illustrated by the example of the preceding paragraph. When scale A was used to measure the position of line (1), the value 3 was ascertained by the experimenter to represent his judgment of the reading on his scale, but the actual value could be 2.5 or 2 or 3.3 or 4, although values less than 2 or more than 4 would seem unlikely. Thus the error in the value 3 units is about 1 cm either direction, so we write 3 ± 1 to indicate our uncertainty. Similarly in the uses of scales B and C with the same line we might write 3.3 ± .1 and 3.35 ± .02 respectively, depending on how accurately we believe we can read the scale.

The errors just discussed are often overshadowed by other errors, caused by a collection of uncontrolled factors such as vibrations of instruments, air currents, temperature fluctuations, etc. For example we might need to measure the length of an object which is longer than our ruler, or a length along a curved surface. To accomplish this we might use a piece of string stretched between the points desired, and then use the ruler to measure the string. When we do this we have no assurance that the string was put under the same tension each time. We thus are led to repeating the measurement several times. The fact that we get slightly different results each time indicates that the uncontrolled factors are present. In this case statistics indicates that the best estimate of the true length or distance is often the average or arithmetic mean of the results of the various trials:

$$Average = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \;, \tag{1}$$

where the x_i are the individual measurements and n is the total number of measurements. (Each $\bar{x}$ will be stated to a number of significant figures given by the accuracy limit of the apparatus. The average $\bar{x}$ should be rounded to the same accuracy.) The experimental values will differ from the average by an amount (x_i - $\bar{x}$) called the deviation of the measurement from x_i. A simple measure of the error in the measurements is given by the average deviation defined as the average of $| x_i - \bar{x} |$ (i. e. the deviations, all taken as positive):

$$\textit{Average deviation}, \text{ a.d.} = \frac{1}{n}\sum_{i=1}^{n} | x_i - \bar{x} | . \tag{2}$$

Note: The average deviation is actually a rather crude estimate of the error in the measurements. According to statistical theory a better measure is given by the standard deviation, defined by:

$$\textit{Standard deviation} = \sigma_x = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2} \;. \tag{3}$$

However since σ_x is harder to compute than the *average deviation* (a.d.), and usually does not differ from it by more than about 25%, in this laboratory we will be satisfied to use the average deviations. Some pocket calculators have the σ_x function built in. If you have access to such a calculator you may quote the σ_x in place of the *average deviation* in all labs except this lab.

While the *average deviation* (a.d.) gives an estimate of the accuracy of individual measurements, we still have not assigned an error to $\bar{x}$. If we take a large number of measurements we would expect $\bar{x}$ to be very close to the true value, while the *average deviation* should not be very different from the result with only a few measurements. Statistical theory tells us that the best error to assign to the mean $\bar{x}$ is the average deviation of the mean (a.d.m.) defined by

$$a.d.m. = \frac{1}{\sqrt{n}} a.d., \tag{4}$$

where n is the total number of measurements. Thus as n increases, while the a.d. does not vary much, the a.d.m. decreases and the faith we would place in $\bar{x}$ becomes greater. However, the ultimate accuracy in $\bar{x}$ is still limited, of course, by the number of significant figures in the measurements, so there is no point in letting n get to be too large.

As an example, suppose we make four measurements of the length of an object, with the results below (in meters):

measured values (x_i)	deviations (x_i - $\bar{x}$)
14.9	.3$\underline{5}$
14.4	-.1$\underline{5}$
14.1	-.4$\underline{5}$
$\underline{14.8}$	$\underline{.25}$
4)$\underline{58.20}$	a.d = $\frac{(.3\underline{5} + .1\underline{5} + .4\underline{5} + .2\underline{5})}{4} = \frac{1.\underline{2}0}{4} = .3\underline{0}$
$\bar{x}$ =14.5$\underline{5}$ m	a.d.m. = .3$\underline{0}$ /$\sqrt{4}$ = .1$\underline{5}$

Note that we have kept one extra figure beyond the significant figures in x_i when computing $\bar{x}$, the deviations, the a.d. and the a.d.m. But it would not be proper as a final result to express the measured length as 14.5$\underline{5}$ ± .1$\underline{5}$ m because there are not this many significant figures possible. The result should be rounded to 14.6 ± .2 m. If many more measurements were made the a.d. would remain .3$\underline{0}$ or thereabouts, while the a.d.m. would decrease. But after 9 measurements the a.d.m. would be less than .1 and further improvements would not be realistic because this is the limit of the instrument as indicated by the significant figures. Note also that in computing the a.d. we took the absolute value of all the deviations i.e. treated them as if positive.

It should be noted that these techniques are only valid if the experiment can be repeated a number of times and if the errors are truly random. Not included in this analysis are systematic error i.e. unknown and uncontrolled biases, which shift all the measurements off in the same direction. An example would be a wooden ruler, which was improperly calibrated or had shrunk. Nor are we including blunders e.g. reading the rule improperly (e.g. 13.6 for 14.6 m). Systematic errors are particularly difficult to detect and eliminate, and many seemingly fine experiments have proved to be erroneous because of some unsuspected systematic bias in the procedure.

Error Propagation in Calculations

Suppose that the length measured in the preceding paragraph, a = 14.6 ± .2 [m], is one side of a rectangle, the other side of which is b = 6.34 ± .02 [m]. From these two numbers we can compute the perimeter of the rectangle $P = 2(a + b)$ or its area $A = ab$. The question arises: how do we compute P and A with our uncertain values of a and b, and what errors do we assign to the computed values?

Let us look first at P. If we use the mean values for a and b we find P = 2 (14.6 + 6.34) = 41.88 [m]. Statistical theory indeed says that this is the best value for P, provided that the errors in a and b are not correlated. (As we saw earlier, the 8 in the hundredths place is not significant since a was accurate only to the tenths place, so we should round this value to 41.9 [m].) With regard to the error to be assigned to this value, we could take the maximum values a = 14.6 + .2 = 14.8, b = 6.34 + .02 = 6.36 and yield P = 42.32 [m] and take minimum values a = 14.6 - .2 = 14.4, b = 6.34 - .02 = 6.32 and yield P = 41.44 [m]. The error is thus ± .44 [m] and P = 41.88 ± .44 [m], or rounding, P = 41.9 ± .4 [m]. The uncertainty in P is thus $2(\Delta a + \Delta b)$ where Δa and Δb are the uncertainties in $\bar{a}$ and $\bar{b}$. We are thus lead to the simple rule: The error assigned to $\bar{a} + \bar{b}$ is $\Delta a + \Delta b$. And the error assigned to $n\bar{a}$ (where n is an exact quantity like the 2 in the formula for P) is $|n|\Delta a$. Try to convince yourself that the error assigned to $-\bar{a}$ is Δa (that is why we have placed the absolute value sign on n) and thus that the error assigned to $a - b$ is $\Delta a + \Delta b$. We always add the errors, never subtract them. Note: statistical theory shows that this estimate of the error in $a \pm b$ is too crude since it is based upon both values of a and b being at their most extreme possible values, which is unlikely. The error $\Delta a + \Delta b$ is likely to be an overestimate, and a more reasonable value for the error in $a \pm b$ is

$$\sqrt{(\Delta a)^2 + (\Delta b)^2}\,. \tag{5}$$

Now consider the area $A = ab$. Again statistical theory says that the best value to be assigned to A is the product of the mean values $\bar{a}\bar{b}$. What error shall we assign to this? Taking our example A = (14.6)(6.34) = 92.564 [m^2]. As we saw earlier, in a product the number of significant figures is the same as that in the factor with the least number of significant figures. In our case both factors have three significant figures so we should round A to the same accuracy: A = 92.6 [m^2]. Instead of calculating the error in A numerically let us do it symbolically: $(\bar{a} \pm \Delta a)(\bar{b} \pm \Delta b) = \bar{a}\bar{b} \pm \bar{a}\Delta b \pm \bar{b}\Delta a \pm \Delta a \Delta b$. Generally the last term is so small that it may be dropped. We thus see that the possible range of values of A is from $\bar{a}\bar{b} - (\bar{a}\Delta b + \bar{b}\Delta a)$ to $\bar{a}\bar{b} + (\bar{a}\Delta b + \bar{b}\Delta a)$. We thus assign to A the error $\Delta A = (\bar{a}\Delta b + \bar{b}\Delta a)$. There is a better way of expressing this: use the fractional or relative error. If we divide ΔA by the assigned value of A (= ab) we get

$$\frac{\Delta A}{\bar{a}\bar{b}} = \frac{\Delta a}{\bar{a}} + \frac{\Delta b}{\bar{b}}\,. \tag{6}$$

The fractional error in A is the sum of the fractional errors in a and b. In our numerical example the fractional error in a is .2/14.6 = .014 (or 1.4 %), and in b is .02/6.34 = .003 (or .3%). The fractional error in $A = ab$ is thus .014 + .003 = .017 or 1.7%. The absolute error in A is thus ΔA = (.017)(92.6) = 1.6 [m^2], rounding off to one decimal place. Thus we get A = 92.6 ± 1.6 [m^2] or 93 ± 2 [m^2].

By arguments such as these we arrive at certain rules, which we summarize in the next page:

Rule 1: When the result of a calculation involves the addition or subtraction of inexact numbers, the uncertainty in the result is the sum of the uncertainties of the individual terms. If $R = A \pm B \pm C$ ... then

$$\Delta R = \sqrt{(\Delta A)^2 + (\Delta B)^2 + (\Delta C)^2 + \ldots} \leq \Delta A + \Delta B + \Delta C + \ldots. \tag{7}$$

Remember that the last expression is just an approximation.

Rule 2: When the result R involves the product or quotient of inexact numbers, the fractional uncertainty in R is calculated as the following:

$$\text{if } R = \frac{AB}{C},\ \frac{\Delta R}{R} = \sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2 + \left(\frac{\Delta C}{C}\right)^2} \leq \frac{\Delta A}{|A|} + \frac{\Delta B}{|B|} + \frac{\Delta C}{|C|}. \tag{8}$$

(The rule for the uncertainty in na, where n is exact, follows from Rule 2 with Δn equal to zero.)

Rule 3: When R involves powers, the fractional uncertainty in R is calculated as below:

$$\text{if } R = \frac{A^p B^q}{C^r}, \text{ then } \frac{\Delta R}{R} = \sqrt{\left(p\frac{\Delta A}{A}\right)^2 + \left(q\frac{\Delta B}{B}\right)^2 + \left(r\frac{\Delta C}{C}\right)^2} \leq p\frac{\Delta A}{|A|} + q\frac{\Delta B}{|B|} + r\frac{\Delta C}{|C|}. \tag{9}$$

Rule 4: When R involves trig functions of an inexact argument we use the following rules (ΔA in rad):

$$\text{If } R = \sin A, \text{ then } \Delta R = |\cos A|\ \Delta A. \tag{10}$$

$$\text{If } R = \cos A, \text{ then } \Delta R = |\sin A|\ \Delta A. \tag{11}$$

Rule 5: Errors or uncertainties (ΔA, etc.) are always treated as positive.

Rule 6: Always round off final results to the appropriate number of significant figures.

We conclude with two comprehensive examples.

Example 1: Suppose $R = (A + BC)/D$

Let $R = R_N/R_3 = (R_1 + R_2)/R_3$, where $R_N = R_1 + R_2$, $R_1 = A$, $R_2 = BC$, and $R_3 = D$.

First of all, $\Delta R_1 = \Delta A$ and $\Delta R_3 = \Delta D$. And using rule 2, $\dfrac{\Delta R_2}{R_2} = \dfrac{\Delta R_2}{BC} = \sqrt{\left(\dfrac{\Delta B}{B}\right)^2 + \left(\dfrac{\Delta C}{C}\right)^2}$.

Then, $\dfrac{\Delta R}{R} = \sqrt{\left(\dfrac{\Delta R_N}{R_N}\right)^2 + \left(\dfrac{\Delta R_3}{R_3}\right)^2}$, where $\Delta R_N = \sqrt{(\Delta R_1)^2 + (\Delta R_2)^2}$.

Example 2: Suppose $R = (2A^3 \sin B)/\sqrt{C}$

Let $R_1 = 2A^3/\sqrt{C}$ and $R_2 = \sin B$, so $R = R_1R_2$. Then by Rule 2, $\dfrac{\Delta R}{R} = \sqrt{\left(\dfrac{\Delta R_1}{R_1}\right)^2 + \left(\dfrac{\Delta R_2}{R_2}\right)^2}$.

By Rules 2 and 3, we find $\dfrac{\Delta R_1}{R_1} = \sqrt{\left(3\dfrac{\Delta A}{A}\right)^2 + \left(\dfrac{1}{2}\dfrac{\Delta C}{C}\right)^2}$,

while by Rule 4, $\Delta R_2 = |\cos B|\Delta B$, or $\dfrac{\Delta R_2}{R_2} = \dfrac{|\cos B|\Delta B}{|\sin B|} = |\cot B|\Delta B$.

Note that the factor of 2 does not enter the formula for the fractional error. When we multiply through by R to get the absolute error in R, ΔR, the 2 will enter properly.