PROBLEM 3

3.1 Principle of least action: Every mechanical system is characterized by a definite function
\[ L = L(q_1, \ldots, q_s, \dot{q}_1, \ldots, \dot{q}_s, t) \]
and the motion of the system is such that the system moves between two fixed positions at different times \( t_1 \) and \( t_2 \) in a way that for sufficiently short time differences the integral
\[ S = \int_{t_1}^{t_2} L \, dt \]
takes the least possible value.

3.2 Deviation of the Euler-Lagrange equations from the least action principle in general coordinates \( q_i, \dot{q}_i, i = 1, \ldots, s \):
\[
0 = \delta \int_{t_1}^{t_2} dt \{ L(q_i + \delta q_i, \dot{q}_i + \delta \dot{q}_i, t) - L(q_i, \dot{q}_i, t) \} = \int_{t_1}^{t_2} dt \left\{ \sum_i \frac{\partial L}{\partial q_i} \delta q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i - L(q_i, \dot{q}_i, t) \right\} = \int_{t_1}^{t_2} dt \left\{ \sum_i \frac{\partial L}{\partial q_i} \delta q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \delta q_i \right\} = \int_{t_1}^{t_2} dt \sum_i \left\{ \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right\} \delta q_i.
\]
The last equality holds because of \( \delta q_i(t_1) = \delta q_i(t_2) = 0 \). As the variations are independent, the finally obtained relation is equivalent to
\[
\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0 \quad \text{for} \quad i = 1, \ldots, s.
\]

3.3 \( L(q_k + \epsilon_k, \dot{q}_k, t) = L(q_k, \dot{q}_k, t) \) implies
\[
0 = \frac{\partial L}{\partial q_k} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} \quad \Rightarrow \quad p_k = \frac{\partial L}{\partial \dot{q}_k} \quad \text{conserved (generalized momentum)}.
\]