PROBLEM 1

(1) The momentum is
\[ p = \frac{\partial L}{\partial \dot{x}} = m \dot{x} \]

(2) Eliminating \( \dot{x} \) in favor of \( p \) we obtain the Hamiltonian
\[ H = T + V = \frac{p^2}{2m} + \frac{1}{2} kx^2 \]

(3) Hamilton’s equations are
\[ \frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x} \quad \text{and} \quad \frac{\partial H}{\partial x} = kx = -\dot{p} \]

(4) Newton’s force law follows
\[ kx = -\dot{p} = -m \ddot{x} \quad \text{or} \quad m \ddot{x} = -kx \]

PROBLEM 2

We consider an infinitesimal rotation by an angle \( \delta \vec{\phi} \) (Landau-Lifshitz, §9, p.18):
\[ \delta \vec{r}_j = \delta \vec{\phi} \times \vec{r}_j, \quad \delta \dot{\vec{r}}_j = \delta \vec{\phi} \times \dot{\vec{r}}_j. \quad (1) \]

For example, if this rotation is about the \( z \) axis \( |\delta \vec{r}_j| = |\delta \phi r_j \sin(\theta)| \) holds. Let us denote the components of the rotations by \( \delta x^i_j \) and \( \delta \dot{x}^i_j \), where \( i = 1, 2, 3 \) labels the coordinates and \( j = 1, \ldots, n \) the particles. Assuming isotropy of space, the Lagrangian is invariant under an infinitesimal rotation
\[ 0 = \sum_j \left\{ \sum_i \frac{\partial L}{\partial x^i_j} \delta x^i_j + \sum_i \frac{\partial L}{\partial \dot{x}^i_j} \delta \dot{x}^i_j \right\}. \]

Using the definition of the generalized momentum and Euler-Lagrange equations, this reads
\[ 0 = \sum_j \left\{ \sum_i \dot{p}^i_j \delta x^i_j + \sum_i p^i_j \delta \dot{x}^i_j \right\} = \sum_j \left\{ \dot{\vec{p}}_j \cdot \delta \vec{r}_j + \vec{p}_j \cdot \delta \dot{\vec{r}}_j \right\}. \]

Inserting (1)
\[ 0 = \sum_j \left\{ \dot{\vec{p}}_j \cdot (\delta \vec{\phi} \times \vec{r}_j) + \vec{p}_j \cdot (\delta \vec{\phi} \times \dot{\vec{r}}_j) \right\} \]

and we pull out the \( \delta \vec{\phi} \), which is the same for all particles:
\[ 0 = \sum_j \left\{ \delta \vec{\phi} \cdot (\vec{r}_j \times \dot{\vec{p}}_j) + \delta \vec{\phi} \cdot (\dot{\vec{r}}_j \times \vec{p}_j) \right\} = \delta \vec{\phi} \frac{d}{dt} \sum_j (\vec{r}_j \times \vec{p}_j) \]

\[ \Leftrightarrow \sum_j (\vec{r}_j \times \vec{p}_j) = \vec{L} = \text{Constant}. \]
PROBLEM 3: See Homework, Problem 19.

PROBLEM 4: Compare Homework, Problem 29.

1. Given a coordinate system \((x', y')\) which rotates with the disk, the location of the CM is \(\bar{x}_{CM} = 0\) and

\[
\bar{y}_{CM} = \frac{\rho}{M} \left\{ \int_{0}^{R} dr r \int_{0}^{\pi} d\theta r \sin \theta + 2 \int_{0}^{R} dr r \int_{\pi}^{2\pi} d\theta r \sin \theta \right\} = \frac{2 \rho R^3}{3 M} = -\frac{4R}{9\pi} \quad \text{as} \quad M = \frac{3\pi \rho R^2}{2}.
\]

2. The relationship between the lab coordinates and the CM coordinates is

\[
x_{CM} = R\theta - |\bar{y}_{CM}| \sin \theta = R\theta - \left(\frac{4R}{9\pi}\right) \sin \theta,
\]

\[
y_{CM} = R - |\bar{y}_{CM}| \cos \theta = R - \left(\frac{4R}{9\pi}\right) \cos \theta.
\]

3. To find \(I_{CM}^{3}\) we calculate first \(I_0^3\) with respect to the center of disk and use the parallel axis theorem, (32.12) of Landau and Lifshitz,

\[
I_{0}^3 = \rho \int_{0}^{R} dr r \int_{0}^{\pi} d\theta (x^2 + y^2) + 2 \rho \int_{0}^{R} dr r \int_{\pi}^{2\pi} d\theta (x^2 + y^2)
\]

\[
= \rho \frac{R^4}{4} + 2 \rho \frac{R^4}{4} = \rho R^4 \frac{3}{4} \pi = \frac{1}{2} MR^2
\]

\[I_{CM}^3 = I_0^3 - M\bar{y}_{CM}^2 = \frac{1}{2} MR^2 - M \frac{16 R^2}{81 \pi^2} = \frac{1}{2} MR^2 \left[1 - \frac{32}{81\pi^2}\right].\]

4. The Lagrangian of the disk is

\[
L = \frac{1}{2} M (\dot{x}_{CM}^2 + \dot{y}_{CM}^2) + \frac{1}{2} I_3 \dot{\theta}^2 - M g y_{CM}.
\]