Always remember to write full work for what you do. This will help your grade in case of incomplete or wrong answers. Also, no credit will be given for an answer, even if correct, if you give no justification for it.

Write your final answers on the sheets provided. You may separate them as long as you put your name on each of them. We will staple them when you hand them in. Ask if you need extra sheets, they will be provided. Remember to put your name on each of them and add them after the problem they refer to.
A particle of mass $m$ is attached to a spring with equilibrium length $a$ and spring constant $k$. The other extrema of the spring is attached to a fixed pivot that you can take as the origin of your coordinate system. There is no gravitational force.

(1.a) What is the force acting on the mass $m$? What is the corresponding potential energy?

(1.b) Is the particle angular momentum $\vec{L}$ conserved? Justify your answer. Show that if $\vec{L} = l\hat{z}$ then the particle moves in the $(x, y)$ plane.

(1.c) Write down the Lagrangian of the particle $m$ and express it using planar polar coordinates $(r, \theta)$.

(1.d) Derive the Euler-Lagrange equations of motion for $r$ and $\theta$ and show that you can reduce them to a single equation which describes the motion of an effective 1-dimensional system (with coordinate $r$ and velocity $\dot{r}$).

(1.e) Determine the effective potential and sketch it as a function of $r$. Describe the possible orbits and show that the system admits a circular orbit.

(1.f) Find the frequency of small radial oscillations about $r = r_0$, the radius of the circular orbit.

(1.g) Sketch the orbit of the particle in the case that the frequency of small radial oscillations is i) twice the angular frequency, ii) equal to the angular frequency, iii) half of the angular frequency.
(1.a) \[ \vec{F}(r) = -k(r-a)\hat{r} \implies V(r) = \frac{1}{2}k(r-a)^2. \]

(1.b) We have \[ \dot{\vec{L}} = \vec{N}, \]
where \( \vec{N} \) is the torque. Since \( \vec{N} = 0 \) for a central force, \( \dot{\vec{L}} = 0 \) and \( \vec{L} \) is therefore conserved. Now since \[ \vec{L} = m\vec{r} \times \dot{\vec{r}}, \]
\( \vec{L} \) is orthogonal to the plane defined by \( \vec{r} \) and \( \dot{\vec{r}} \), i.e. to the plane of motion. Since \( \vec{L} \) is constant, this motion is planar. If \( L = l\hat{z} \), the motion is therefore in the \((x, y)\) plane.

(1.c) The Lagrangian is \[ L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}k(r-a)^2. \]

(1.d) The equation of motion for \( r \) gives us \[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = 0 \implies \frac{d}{dt}(m\ddot{r}) - m\dot{\theta}^2 + k(r-a) = 0 \]
\[ \implies m(\ddot{r} - r\dot{\theta}^2) = -k(r-a). \]
This reads \( ma_r = F_r \), which is the radial component of \( m\ddot{r} = \vec{F} \). Now for the equation of motion in \( \theta \):
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \implies \frac{d}{dt}(mr^2\dot{\theta}) = 0 \]
\[ \implies mr(r\dot{\theta} + 2\dot{r}\dot{\theta}) = 0 \implies ma_\theta = 0. \]
Therefore \[ \frac{d}{dt}(l) = 0, \]
where the magnitude of the angular momentum is \( l = mr^2\dot{\theta} \). So the \( (\theta) \) equation expresses the conservation of angular momentum, as we could expect from the fact that \( \theta \) is cyclic.

Using \( \dot{\theta} = \frac{l}{mr^2} \) in the \( (r) \) equation we get \[ m\ddot{r} = \frac{l^2}{mr^3} - k(r-a) = F'(r), \]
which is the equation of a 1-dimensional “effective” system described by a coordinate \( r \).
From the expression of $F'(r)$ we can define our “effective” potential $V'(r)$ such that

$$F'(r) = -\frac{dV'(r)}{dr},$$

where

$$V'(r) = \frac{1}{2} \frac{l^2}{mr^2} + \frac{1}{2} k(r - a)^2.$$

See Figure (1) for a graph of this potential.

There are three cases for the description of the orbits:

- $E < E_0 \rightarrow$ unphysical
- $E = E_0 \rightarrow$ circular orbit of $r = r_0$ where $r_0$ is the solution of
  $$\frac{dV'(r)}{dr} = 0 \Rightarrow -\frac{l^2}{mr^3_0} + k(r_0 - a) = 0.$$
- $E > E_0 \rightarrow$ bounded orbits with $r_1 \leq r \leq r_2$.

Start with $r = r_0 + \delta$, and we find

$$m \ddot{\delta} = \frac{l^2}{m(r_0 + \delta)^3} - k(r_0 + \delta - a)$$

$$= \frac{l^2}{mr^3_0} \left(1 - \frac{3\delta}{r_0} + o(\delta^2)\right) - k(r_0 - a) - k\delta$$

$$= \frac{l^2}{mr^3_0} - k(r_0 - a) - \left[\frac{3l^2}{mr^4_0} + k\right] \delta$$

The first two terms on the right are zero from the condition

$$\left. \frac{dV'(r)}{dr} \right|_{r=r_0} = 0.$$

Thus we have

$$\ddot{\delta} + \left[\frac{3l^2}{m^2r^4_0} + \frac{k}{m}\right] \delta = 0 \rightarrow \ddot{\delta} + \Omega^2 \delta = 0.$$

This tells us that frequency of small oscillations about $r = r_0$ is

$$\Omega = \left[\frac{3l^2}{m^2r^4_0} + \frac{k}{m}\right]^{1/2}. $$

The same result could be obtained by using that

$$\Omega = \left[\frac{1}{m} \left. \frac{\partial^2 V'(r)}{\partial^2 r} \right|_{r=r_0}\right]^{1/2}.$$
Figure 1: A plot of the effective potential. The dimensionless radial variable is $\eta = \alpha r$ with $\alpha = \sqrt{2m/l}$, and $\beta^2 = kl^2/4m$. We have chosen the specific case of $\alpha = 1/a$ and $\beta = 3$ for convenience.

Figure 2: Three possible orbital configurations

(1.g)

We have three cases here (see Figure 2):

- $\Omega = 2\dot{\theta}_0 = \frac{l}{m\dot{r}_0}$, then the orbit oscillates twice between $r_1$ and $r_2$ for each revolution (Figure 2i).
- $\Omega = \dot{\theta}_0$, then the orbit oscillates once (Figure 2ii).
- $\Omega = \frac{1}{2}\dot{\theta}_0$, then the orbit takes 2 revolutions to oscillate between $r_1$ and $r_2$ (Figure 2iii).
Problem 2

Consider a uniform disk of mass $m$ and radius $a$.

(2.a) Show that the principal moments of inertia $I_i$ for rotation about a set of (principal) axes $\{x_1, x_2, x_3\}$ through the center of mass (CM) of the disk (see figure) are $I_1 = I_2 = ma^2/4$ and $I_3 = ma^2/2$.

(2.b) Find the principal moments of inertia for a set of axes $\{X_1, X_2, X_3\}$ parallel to the axes in (2.a) but relative to an origin $O$ at the edge of the disk (see figure).

(2.c) Imagine the disk is now suspended from its edge and rotates about that suspension point in a plane parallel to the disk, under the action of the force of gravity. Find the Lagrangian of the disk in terms of the angle $\theta$ and its associated angular velocity $\dot{\theta}$.

(2.d) Find the Euler-Lagrange equations of motion for $\theta$.

(2.e) Find the frequency of small oscillations of the disk around its equilibrium point for such rotations.
(2.a)

\[ I_1 = \rho \int_0^a dr \int_0^{2\pi} d\theta \left( y^2 + z^2 \right) = \rho \int_0^a dr \int_0^{2\pi} d\theta \sin^2 \theta \]

\[ = \rho \frac{a^4}{4\pi} = \frac{ma^2}{\pi a^2 \frac{a^4}{4}} = \frac{ma^2}{4} \]

\[ I_2 = I_1 \]

\[ I_3 = \rho \int_0^a dr \int_0^{2\pi} d\theta \left( x^2 + y^2 \right) = 2I_1 = \frac{ma^2}{2} \]

(2.b)

We use Steiner’s theorem:

\[ J_{ij} = I_{ij} + m \left[ a^2 \delta_{ij} - a_i a_j \right] \]

where \( \vec{a} = (0, a, 0) \). Then

\[ J_1 = I_1 + ma^2 = \frac{ma^2}{4} + ma^2 = \frac{5}{4} ma^2 \]

\[ J_2 = I_2 + m(a^2 - a^2) = I_2 = \frac{ma^2}{4} \]

\[ J_3 = I_3 + ma^2 = \frac{ma^2}{2} + ma^2 = \frac{3}{2} ma^2 \]

and the other components are zero.
Using coordinates:

\[
\begin{align*}
  x_{CM} &= a \sin \theta \\
  y_{CM} &= -a \cos \theta
\end{align*}
\]

\[
\begin{align*}
  \dot{x}_{CM} &= a \dot{\theta} \cos \theta \\
  \dot{y}_{CM} &= a \dot{\theta} \sin \theta
\end{align*}
\]

We have two ways to calculate \( T \):

- Using 0 as the origin, in which case the kinetic energy is purely rotational because 0 is fixed (i.e. is the center of rotation, or, the axis of rotation goes through 0). Then
  \[
  T = \frac{1}{2} J_3 \dot{\theta}^2 = \frac{3}{2} 2 \frac{ma^2 \dot{\theta}^2}{2} = \frac{3}{4} ma^2 \dot{\theta}^2.
  \]

- Using the CM as the origin, in which case the kinetic energy is the sum of \( T_{CM} + T_{rot} \) (with respect to the CM). i.e.,
  \[
  T = \frac{1}{2} m (\dot{x}_{CM}^2 + \dot{y}_{CM}^2) + \frac{1}{2} I_3 \dot{\theta}^2
  = \frac{1}{2} ma^2 \dot{\theta}^2 + \frac{1}{2} \frac{1}{2} ma^2 \dot{\theta}^2 = \frac{3}{4} ma^2 \dot{\theta}^2.
  \]

The potential energy is

\[
U = -mga \cos \theta,
\]

and the Lagrangian is

\[
L = T - U = \frac{3}{4} ma^2 \dot{\theta}^2 + mga \cos \theta.
\]

\(2.d\)

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{3}{2} ma^2 \ddot{\theta} + mga \sin \theta = 0
\]

\[
\Rightarrow \ddot{\theta} + \frac{2g}{3a} \sin \theta = 0.
\]
For small angles, 
\[ \sin \theta \approx \theta + o(\theta^3), \]
and the equation of motion becomes
\[ \ddot{\theta} + \frac{2g}{3a} \theta = 0. \]
Thus the frequency of small oscillations is
\[ \omega = \sqrt{\frac{2g}{3a}}. \]
Problem 3

A rigid body is in the shape of a thin rectangular sheet of sides $2b$ and $2c$, as shown. The sheet has mass $M$, and the mass is uniformly distributed over it. The body is rotating with constant angular velocity $\omega$.

(3.a) Find the angular velocity $\vec{\omega}$ in the body coordinate system made up of the principal axes through the center of mass (CM).

Consider the system of principal axes with origin at the center of the rectangular sheet and axes parallel to the sides of the sheet (see figure). In this system the angular velocity can be written as,

$$\vec{\omega} = (\omega \cos \alpha, \omega \sin \alpha, 0) = \left( \frac{\omega b}{\sqrt{b^2 + c^2}}, \frac{\omega c}{\sqrt{b^2 + c^2}} \right) = \frac{\omega}{\sqrt{b^2 + c^2}} (b, c, 0) ,$$

where $\omega = |\vec{\omega}|$, $\alpha$ is the angle between the $x_1$ axis and $\vec{\omega}$, $\cos \alpha = b/\sqrt{b^2 + c^2}$ and $\sin \alpha = c/\sqrt{b^2 + c^2}$.

(3.b) Find the angular momentum $\vec{L}$ in the body coordinate system. The principal moments of inertia are $I_1 = \frac{1}{3} Mc^2$ (axis parallel to side of length $2b$), $I_2 = \frac{1}{3} Mb^2$ (axis parallel to side of length $2c$), and $I_3 = \frac{1}{3} M(b^2 + c^2)$ (axis orthogonal to the plane of the rectangular sheet).

The angular momentum is

$$\vec{L} = (I_1 \omega_1, I_2 \omega_2, I_3 \omega_3) = \frac{1}{3} \frac{mbc\omega}{\sqrt{b^2 + c^2}} (c, b, 0) .$$
(3.c) Why is a torque $\vec{N}$ required to maintain this angular velocity? Find it in the body frame.

The torque acting on the body is,

$$\vec{N} = \left( \frac{d\vec{L}}{dt} \right)_{\text{fixed}} = \left( \frac{d\vec{L}}{dt} \right)_{\text{body}} + \vec{\omega} \times \vec{L} .$$

From the expression of $\vec{L}$ derived in (f1.b) we see that,

$$\left( \frac{d\vec{L}}{dt} \right)_{\text{body}} = 0 ,$$

and therefore a torque,

$$\vec{N} = \vec{\omega} \times \vec{L} = (\omega_1 L_2 - \omega_2 L_1) \hat{e}_3 = \frac{1}{3} \frac{mbc(b^2 - c^2)\omega^2}{(b^2 + c^2)} \hat{e}_3 ,$$

where $\hat{e}_3$ is the unit vector in the direction orthogonal to the sheet, is required to maintain the angular velocity $\vec{\omega}$.

(3.d) Describe what happens to $\vec{\omega}$, $\vec{L}$, and $\vec{N}$ as viewed from a fixed frame through the CM.

$\vec{\omega}$ is constant in both frames, since

$$\left( \frac{d\vec{\omega}}{dt} \right)_{\text{fixed}} = \left( \frac{d\vec{\omega}}{dt} \right)_{\text{body}} = 0 ,$$

while $\vec{L}$ precesses about the direction of $\vec{\omega}$ with precession rate $\omega$, since

$$\left( \frac{d\vec{L}}{dt} \right)_{\text{fixed}} = \left( \frac{d\vec{L}}{dt} \right)_{\text{body}} \times \vec{\omega} \times \vec{L} = \vec{\omega} \times \vec{L} .$$

The same applies to $\vec{N}$: $\vec{N}$ precesses about $\vec{\omega}$ keeping orthogonal to $\vec{\omega}$ and $\vec{L}$.