Problem 1

Consider the problem of an isolated particle of mass \( m \) moving freely in a one-dimensional box of size \( L \). Such a particle satisfies the following one-dimensional Schrödinger equation:

\[
-\frac{\hbar^2}{2m} \frac{d^2 \varphi(x)}{dx^2} = \epsilon \varphi(x) , \quad \text{with} \quad \varphi(0) = \varphi(L) = 0 .
\]

1.a) Obtain the eigenvalues and the normalized eigenvectors of the Schrödinger equation.

Now consider an isolated system of 4 non-interacting particles of mass \( m \) placed in such one-dimensional box. The energy of the system equals \( E = 63\epsilon_0 \), where \( \epsilon_0 = \frac{\hbar^2 \pi^2}{2mL^2} \), is the lowest eigenvalue of the Schrödinger equation. Find the entropy of the system for the following cases:

1.b) a system of 4 distinguishable spinless particles;

1.c) a system of 4 indistinguishable spinless bosons;

1.d) a system of 4 indistinguishable spin-1/2 fermions.

Problem 2

Consider a system of \( N \) localized particles moving under the influence of a quantum, one-dimensional, harmonic-oscillator potential of frequency \( \omega \). The energy of the system is given by

\[
E = \frac{1}{2} N\hbar\omega + M\hbar\omega ,
\]

where \( M \) is the total number of quanta in the system. That is,

\[
M = \sum_{i=1}^{N} n_i ,
\]

with \( n_i = 0, 1, 2, \ldots \) representing the number of quanta in the \( i_{th} \) harmonic oscillator.

2.a) Compute the number of microstates \( \Gamma \) as a function of \( N \) and \( M \).
2.b) Using Stirling’s approximation, compute the entropy of the system as a function of \( N \) and \( M \).

2.c) Compute the temperature \( T \) of the system as a function of \( N \) and \( M \). Are there any values of \( N \) and \( M \) for which the temperature \( T \) becomes negative?

2.d) Compute the specific heat \( C_V \) of the system as a function of \( N \) and \( T \). Note that in order to do so, you will need to express the total energy of the system as a function of \( N \) and \( T \).

2.e) Compute the low- \( (k_B T \ll \hbar \omega) \) and high-temperature \( (k_B T \gg \hbar \omega) \) limits of the specific heat, and make a simple plot of its behavior as a function of temperature.

**Problem 3**

Problem 1.4 of Pathria’s book.