1 Graded problems

1. 1.a) For a one-dimensional system with Hamiltonian

\[ H = \frac{p^2}{2} - \frac{1}{2q^2}, \]

show that there is a constant of motion given by

\[ D = \frac{pq}{2} - Ht. \]

1.b) As a generalization of part 1.a), show that for a motion in a plane with the Hamiltonian

\[ H = |p|^n - ar^{-n}, \]

where \( p \) is the vector of the momenta conjugate to the Cartesian coordinates, there is a constant of motion given by

\[ D = \frac{p \cdot r}{n} - Ht. \]

1.c) The transformation \( Q = \lambda q, \ p = \lambda P \) is obviously canonical. However, the same transformation with \( t \) time dilatation, \( Q = \lambda q, \ p = \lambda P, \ t' = \lambda^2 t \), is not. Show that, however, the equations of motion for \( q \) and \( p \) for the Hamiltonian in part 1.a) are invariant under this transformation. The constant of motion \( D \) is said to be associated with this invariance.

2. Given a system with Hamiltonian

\[ H = \frac{1}{2} \left( \frac{1}{q^2} + p^2 q^4 \right), \]

2.a) find the equation of motion for \( q \);

2.b) find a canonical transformation that reduces \( H \) to the Hamiltonian of a harmonic oscillator. Show that the solution for the transformed variables is such that the equation of motion found in part 2.a) is satisfied.

3. 3.a) Show that the transformation

\[ Q = p + iaq, \ P = \frac{p - iaq}{2ia}, \]

with \( a \) a real constant, is canonical and find a generating function.

3.b) Use the transformation to solve the linear harmonic oscillator problem.