1 Graded problems

Problem 1

1.a)  
The Lagrangian is

\[ L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - V(r), \]  

and the conjugate momenta are

\[ p_r = \frac{\partial L}{\partial \dot{r}} = m \dot{r}, \]  
\[ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2 \dot{\theta}, \]  
\[ p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin^2 \theta \dot{\phi}. \]  

By integrating these relations we get

\[ \dot{r} = \frac{p_r}{m}, \]  
\[ \dot{\theta} = \frac{p_\theta}{mr^2}, \]  
\[ \dot{\phi} = \frac{p_\phi}{mr^2 \sin^2 \theta}. \]  

Thus the Hamiltonian is

\[ H = p_i \dot{q}_i - L = \frac{1}{2m} \left[ p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta} \right] + V(r). \]  

Using our class discussion and Goldstein § 8.1 we can find

\[ \ddot{a} = 0, \]  
\[ \hat{T} = m \begin{pmatrix} 1 & r^2 & 0 \\ 0 & r^2 \sin^2 \theta & 0 \end{pmatrix} \rightarrow \hat{T}^{-1} = \frac{1}{m} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}, \]  
\[ H = \frac{1}{2} \vec{p}^T \hat{T}^{-1} \vec{p} + V \text{ with } \vec{p} = \begin{pmatrix} p_r \\ p_\theta \\ p_\phi \end{pmatrix}. \]
Thus we see that 
\[ H = T + V = E , \]
as expected. Hamilton’s equations of motion are

\[
\begin{align*}
\begin{pmatrix} r \\ p_r \end{pmatrix} & \quad \begin{cases} \dot{r} = \frac{\partial H}{\partial p_r} - \frac{p_r}{m} = \frac{1}{mr^3} \left[ p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta} \right] - V'(r) \\ \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta}{mr^2} \
\end{cases} \\
\begin{pmatrix} \theta \\ p_\theta \end{pmatrix} & \quad \begin{cases} \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \\ \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2 \cos \theta}{mr^2 \sin^3 \theta} \\
\end{cases} \\
\begin{pmatrix} \phi \\ p_\phi \end{pmatrix} & \quad \begin{cases} \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2 \sin^3 \theta} \\ \dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \\
\end{cases}
\end{align*}
\]

1.b) If \( p_\phi(0) = 0 \) then \( p_\phi(t) = 0 \) for all times (since \( \dot{p}_\phi = 0 \) from the equations of motion). Then the equations of motion become

\[
\begin{align*}
\begin{cases} \dot{r} = \frac{p_r}{m} - V'(r) \\
\dot{p}_r = \frac{p_\theta^2}{mr^3} - V'(r) \quad \begin{cases} \dot{\theta} = \frac{p_\theta}{mr^2} \\ \dot{p}_\theta = 0 \\
\end{cases} \\
\dot{p}_\phi = 0 \\
\end{cases}
\end{align*}
\]

Thus if \( \phi(0) = 0 \to \phi(t) = 0 \) at all times, and the motion is planar (in the \( \phi = 0 \) plane) as we would expect. Given the initial conditions, it will be in the \( \phi = 0 \) plane. The \((r, p_r)\) and \((\theta, p_\theta)\) sets of equations reduce to the usual equations for central-force motion:

\[
\begin{align*}
\begin{cases} \dot{\theta} = \frac{p_\theta}{mr^2} \to p_\theta = mr^2 \dot{\theta}, \text{ conserved.} \\
\dot{p}_\theta = 0 \to \dot{p}_\theta = 0 \to mr(r \ddot{\theta} + 2 \dot{r} \dot{\theta}) = 0 = ma_\theta = 0. \\
\end{cases}
\end{align*}
\]

So we have that \( p_\theta = l \) is the magnitude of the angular momentum, and from \( \dot{p}_\theta = 0 \) we get Newton’s second law in the \( \dot{\theta} \) direction. For the radial part:

\[
\begin{align*}
\begin{cases} \dot{r} = \frac{p_r}{m} \\
\dot{p}_r = \frac{p_\theta^2}{mr^3} - V'(r) = \frac{l^2}{mr^3} - V'(r) . \\
\end{cases}
\end{align*}
\]

Taking a derivative of the first equation and plugging it into the second equation we find

\[
\dot{r} = \frac{\dot{p}_r}{m} = \frac{l^2}{m^2 r^3} - \frac{V'(r)}{m} .
\]

Thus we can write

\[
\dot{p}_r = mr^2 - V'(r) = mr^2 \ddot{\theta} - V'(r) ,
\]

\[
\implies m(\ddot{r} - r \ddot{\theta}^2) = ma_r = -V'(r) = F(r) .
\]

So we find this set of equations gives us Newton’s 2nd law in the radial direction.
For the specific case of $\vec{F}(r) = -\frac{k}{r^2}\vec{r}$, since this is a conservative force with no constraint we already know that

$$H = T + U = E,$$

for constant $E$. From the diagram we see the coordinates are

$$x = r\cos\theta \implies v^2 = r^2 + r^2\dot{\theta}^2. \quad (14)$$

The kinetic and potential energy, and the Lagrangian are

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2), \quad (15)$$

$$U = -\int \left( -\frac{k}{r^2} \right) dr = -\frac{k}{r},$$

$$L = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}. \quad (16)$$

For the conjugate momenta we have

$$\begin{cases} p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \rightarrow \quad \dot{r} = \frac{p_r}{m} \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad \rightarrow \quad \dot{\theta} = \frac{p_\theta}{mr^2} \end{cases} \quad (16)$$

Our Hamiltonian is

$$H = \dot{r}p_r + \dot{\theta}p_\theta - L = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{k}{r} = T + U, \quad (17)$$

and $E$ is conserved since

$$\frac{\partial H}{\partial t} = 0 \rightarrow E = \text{constant}. \quad (18)$$

The Hamiltonian (canonical) equations of motion are

$$\frac{\partial H}{\partial p_r} = \frac{\partial H}{\partial \dot{r}} = -\dot{p}_r, \quad \frac{\partial H}{\partial p_\theta} = -\dot{p}_\theta. \quad (19)$$

From the equations of motion for $(\theta, p_\theta)$ we get;

$$\begin{cases} \frac{\partial H}{\partial p_\theta} &= \dot{\theta} = \frac{p_\theta}{mr^2} \implies p_\theta = mr^2\dot{\theta} = \text{constant}, \\ \frac{\partial H}{\partial \theta} &= -\dot{p}_\theta = 0 \end{cases} \quad (20)$$

which shows the angular momentum is conserved. From the set of equations for $(r, p_r)$, we find

$$\frac{\partial H}{\partial p_r} = \dot{r} = \frac{p_r}{m}, \quad (21)$$

$$\frac{\partial H}{\partial r} = -\dot{p}_r = -\frac{p_\theta^2}{mr^3} + \frac{k}{r^2}$$

$$\implies \dot{p}_r = m\ddot{r} = \frac{p_\theta^2}{mr^3} - \frac{k}{r^2} = mr\ddot{\theta}^2 - \frac{k}{r}. \quad (22)$$
Problem 2

2.a)

Our particle moves in 1D, so we use one generalized coordinate $x$. The potential is given by integrating the force:

$$ F(x, t) = \frac{k}{x^2} e^{-t/\tau} \rightarrow U(x, t) = - \int F(x, t) dx = \frac{k}{x} e^{t/\tau} + C , \quad (23) $$

where we assume as $x \to \infty$ that $U \to 0$ so we take $C = 0$. The kinetic energy and the Lagrangian are therefore

$$ T = \frac{1}{2} m \dot{x}^2 , \quad (24) $$

$$ \Rightarrow L = T - U = \frac{1}{2} m \dot{x}^2 - \frac{k}{x} e^{-t/\tau} . \quad (25) $$

The conjugate momentum of $x$ is

$$ p_x = \frac{\partial L}{\partial \dot{x}} = m \dot{x} \rightarrow \dot{x} = \frac{p_x}{m} . \quad (26) $$

so the Hamiltonian is

$$ H = \dot{x} p_x - L = \frac{1}{2} \frac{p_x^2}{m} + \frac{k}{x} e^{-t/\tau} = T + U = E . \quad (27) $$

2.b)

From above we see $H = E$, since there are no constraints and $U$ is not a function of $\dot{x}$. However, since $U = U(x, t)$ (explicitly depends on time!), the energy of the system is not conserved:

$$ \frac{dE}{dt} = \frac{dH}{dt} = \frac{\partial H}{\partial t} \neq 0 . \quad (28) $$

Problem 3

3.a)

The magnetic field is given to us as $\vec{B}(\vec{r}) = B_0 \hat{z}$, and we can verify that the vector potential $\vec{A}(\vec{r}) = \frac{1}{2} \vec{B} \times \vec{r}$ satisfies $\vec{B} = \vec{\nabla} \times \vec{A}$ in the following way:

$$ (\vec{\nabla} \times \vec{A})_i = \epsilon_{ijk} \partial_j A_k = \epsilon_{ijk} \partial_j \left( \frac{1}{2} \epsilon_{klm} B_l x_m \right) \quad (29) $$

$$ = \frac{1}{2} \epsilon_{ijk} \epsilon_{klm} B_l \delta_{lm} $$

$$ = \frac{1}{2} \epsilon_{ijk} \epsilon_{klj} B_l = \frac{1}{2} 2 \delta_{il} B_l = B_i . $$

This implies exactly that

$$ \vec{A} = \frac{1}{2} B_0 \hat{z} \times \vec{r} = \frac{1}{2} B_0 (x \hat{y} - y \hat{x}) . \quad (30) $$
3.b)

The Lagrangian is

\[
L(\vec{r}, \dot{\vec{r}}) = \frac{1}{2} m \dot{\vec{r}}^2 + \frac{q}{c} \dot{\vec{r}} \cdot \vec{A}(\vec{r})
\]

(31)

\[
= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{c} (\dot{x} A_x + \dot{y} A_y + \dot{z} A_z)
\]

\[
= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q B_0}{2c} (-y \dot{x} + x \dot{y}) .
\]

The conjugate momenta are

\[
p_x = \frac{\partial L}{\partial \dot{x}} = m \dot{x} - \frac{q B_0}{2c} y \rightarrow \dot{x} = \frac{1}{m} \left( \frac{q B_0}{2c} y + p_x \right)
\]

(32)

\[
p_y = \frac{\partial L}{\partial \dot{y}} = m \dot{y} + \frac{q B_0}{2c} x \rightarrow \dot{y} = \frac{1}{m} \left( -\frac{q B_0}{2c} + p_y \right)
\]

\[
p_z = \frac{\partial L}{\partial \dot{z}} = m \dot{z} \rightarrow \dot{z} = \frac{p_z}{m} .
\]

Thus the Hamiltonian is

\[
H = p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - L
\]

(33)

\[
= \frac{q B_0}{2mc} p_x y + \frac{p_x^2}{m} - \frac{q B_0}{2mc} p_y x + \frac{p_y^2}{m} - \frac{1}{2m} (p_x^2 + p_y^2 + p_z^2) - \frac{q B_0}{2mc} \left( -\frac{q B_0}{4c} y^2 - \frac{q B_0}{4c} x^2 \right)
\]

\[
= \frac{1}{2m} (p_x^2 + p_y^2 + p_z^2) - \frac{q B_0}{2mc} \left( -\frac{q B_0}{4c} y^2 - p_x y - \frac{q B_0}{4c} x^2 + p_y x \right)
\]

\[
= \frac{1}{2m} \left( p_x + \frac{q B_0}{2c} y \right)^2 + \frac{1}{2m} \left( p_y - \frac{q B_0}{2c} x \right)^2 + \frac{1}{m} p_z^2 .
\]

3.c)

The mechanical momenta are

\[
\left\{ \begin{array}{l}
\pi_x = m \dot{x} = p_x + \frac{q B_0}{2c} y \\
\pi_y = m \dot{y} = p_y - \frac{q B_0}{2c} x \\
\pi_z = m \dot{z}
\end{array} \right.
\]

(34)

So we have (using Landau’s definition of Poisson’s bracket, as also given in this problem),

\[
\{\pi_x, \pi_y\} = \left\{ p_x + \frac{q B_0}{2c} y, p_y - \frac{q B_0}{2c} x \right\}
\]

(35)

\[
= \{p_x, p_y\} + \frac{q B_0}{2c} \{y, p_y\} - \frac{q B_0}{2c} \{p_x, x\} - \left( \frac{q B_0}{2c} \right)^2 \{y, x\}
\]

\[
= -\left( \frac{q B_0}{2c} \right)^2 2 = -\frac{q B_0}{c} ,
\]

where we have used that \(\{p_x, p_y\} = 0\) and \(\{y, x\} = 0\). Similarly we have

\[
\{\pi_y, \pi_z\} = \left\{ p_y - \frac{q B_0}{2cx}, p_z \right\} = 0 ,
\]

(36)

\[
\{\pi_z, \pi_x\} = \left\{ p_z, p_x + \frac{q B_0}{2c} y \right\} = 0 .
\]
In terms of the mechanical momenta:

\[ H = \frac{\pi_x^2}{2m} + \frac{\pi_y^2}{2m} + \frac{\pi_z^2}{2m}. \] (37)

Now using

\[ \frac{d\vec{\pi}}{dt} = \{H, \vec{\pi}\}, \] (38)

we get

\[ \begin{align*}
\dot{\pi}_x &= \{H, \pi_x\} = \frac{1}{2m}\{\pi_y^2, \pi_x\} = \frac{qB_0}{mc}\pi_y, \\
\dot{\pi}_y &= \{H, \pi_y\} = \frac{1}{2m}\{\pi_x^2, \pi_y\} = -\frac{qB_0}{mc}\pi_x, \\
\dot{\pi}_z &= \{H, \pi_z\} = 0.
\end{align*} \] (39)

The last expression implies \( \pi_z(t) = \text{constant} = \pi_z(0) \). Taking a derivative of the first expression and plugging in the second gives

\[ \ddot{\pi}_x = \frac{qB_0}{mc}\dot{\pi}_y = -\left(\frac{qB_0}{mc}\right)^2 \pi_x = -\omega^2 \pi_x \implies \ddot{\pi}_x + \omega^2 \pi_x. \] (40)

The solution to this is

\[ \pi_x(t) = A \cos(\omega t) + B \sin(\omega t), \] (41)

while for \( \pi_y \) we have

\[ \pi_y(t) = \frac{1}{\omega} (-A \omega \sin(\omega t) + B \omega \cos(\omega t)) = -A \sin(\omega t) + B \cos(\omega t). \] (42)

With the initial conditions

\[ \pi_x(0) = A, \quad \pi_y(0) = B, \] (43)

we can write

\[ \begin{align*}
\pi_x(t) &= \pi_x(0) \cos(\omega t) + \pi_y(0) \sin(\omega t) \\
\pi_y(t) &= -\pi_x(0) \sin(\omega t) + \pi_y(0) \cos(\omega t) \\
\pi_z(t) &= \pi_z(0).
\end{align*} \] (44)

From Newton’s 2nd law we would get

\[ m\ddot{\vec{v}} = m\frac{d\vec{v}}{dt} = \frac{d\vec{\pi}}{dt} = \frac{q}{c}\vec{v} \times \vec{B} = \frac{qB_0}{mc}\vec{\pi} \times \hat{z}. \] (45)

This means that \( \vec{\pi} \) moves precessing about the z-axis with frequency \( \omega = \frac{qB_0}{mc} \), as we have found in the explicit expression for \( \pi_x, \pi_y, \) and \( \pi_z \) above. Also note that, since

\[ \{H, \vec{\pi}\} = \frac{qB_0}{mc}\vec{\pi} \times \hat{z}, \] (46)

we have also found (38).