1. Consider a system of fields $\phi_i(x)$ and a Lorentz transformation acting on both field and coordinates as follows,

$$x'^\mu = \Lambda^\mu_\nu x^\nu \simeq x^\mu + \alpha \sum_{k=1}^6 \alpha_k X^\mu_k + O(\alpha^2),$$

$$\phi'_i(x') = L_{ij}(\Lambda) \phi_j(x) \simeq \phi_i(x) + \alpha \sum_{k=1}^6 \alpha_k A_{ij,k} \phi_j(x) + O(\alpha^2),$$

where $L(\Lambda)$ denote the representation of the Lorentz group on the space of the fields $\phi_i(x)$. Show that if the action $S$,

$$S = \int d^4x \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x), x),$$

associated to the system of fields is invariant under such transformation, there are six conserved currents (Noether’s currents) of the form,

$$M_{\rho \sigma}^\mu = T^\mu_\rho x_\sigma - T^\mu_\sigma x_\rho - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_i(x))} A_{ij,\rho \sigma} \phi_j(x),$$

where $T^{\mu \nu}$ is the energy-momentum tensor associated to the system of fields, and six conserved charges. How are the charges defined? How can you interpret the components of $M_{\rho \sigma}^\mu$ due to the transformation of the coordinates and to the transformation of the fields respectively? Explain your reasoning.

2. Classical electromagnetism (with no sources) follows from the action,

$$S = \int d^4x \left( -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} \right),$$

where $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, as you have derived in your first homework. Construct the energy-momentum tensor for this theory. Notice that the usual procedure does not result in a symmetric tensor. Show that you can remedy that by using a tensor $K^{\lambda \mu \nu}$ antisymmetric in its first two indices (you will have to build a new energy-momentum tensor $\hat{T}^{\mu \nu} = T^{\mu \nu} + \ldots$ where the dots stays for a function of $K^{\lambda \mu \nu}$). In particular show that

$$K^{\lambda \mu \nu} = F^{\mu \lambda} A^\nu,$$

leads to the standard formulae for the electromagnetic energy and momentum densities:

$$E = \frac{1}{2}(E^2 + B^2), \quad S = E \times B.$$
3. Recall that $T(a)^{-1}\phi(x)T(a) = \phi(x-a)$, where $T(a)$ exp$(iP^\mu a_\mu)$ is the spacetime translation operator, and $P^0$ is identified as the hamiltonian $H$.

3.a) Let $a^\mu$ be infinitesimal, and derive an expression for $[\phi(x), P^\mu]$.

3.b) Show that the time component of your result is equivalent to the Heisenberg equation of $i\dot{\phi} = [\phi, H]$.

3.c) For a free field, use the Heisenberg equation to derive the Klein-Gordon equation.

3.d) Define a spatial momentum operator

$$ P = -\int d^3x \Pi(x) \nabla \phi(x) , $$

Use the canonical commutation relations to show that $P$ obeys the relation you derived in part 3.a).

3.e) Express $P$ in terms of $a(k)$ and $a^\dagger(k)$.

4. Problem 2.2 of Peskin and Schroeder’s book.